MATH 507a **QUALIFYING EXAM** **February 1, 2011**

Solve all problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, **and write on only one side of the paper**. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. `This is a warm-up problem on the first Borel-Cantelli lemma.`

Let X_n, $n \geq 1$, be independent (but not necessarily identically distributed) random variables, $S_n = \sum_{k=1}^{n} X_k$, and let a_n be real numbers such that $a_n/a_{n+1} \leq C$ for all n and

$$P\left(\lim_{n\to\infty} \frac{S_n}{a_n} = 0\right) = 1.$$

Show that $\sum_{k\geq 1} P(|X_k| \geq a_k) < \infty$.

Problem 2. `This problem tests you knowledge of the basic properties of the random walk.`

Let $X_1, X_2, \ldots$ be independent and identically distributed, each equal to 1 with probability p and equal to 0 with probability $1-p$. Let $S_n = \sum_{k=1}^{n} X_k$.

1. Prove that if $p \neq \frac{1}{2}$, then, with probability 1, $S_n = 0$ only finitely many times.
2. Prove that if $p = \frac{1}{2}$, then S_n will equal 0 infinitely often, but the mean recurrence time is infinite. In other words, with the notation $\tau = \inf\{n > 1 : S_n = 0\}$, you need to show that $P(\tau < \infty) = 1$ but $E\tau = +\infty$.

Problem 3. `This problem tests your knowledge of the strong law of large numbers.`

(a) Let $X_1, X_2, \ldots$ be independent (but not necessarily identically distributed) random variables such that $\sup_{n\geq 1} E|X_n - EX_n|^4 < \infty$. Define $S_n = \sum_{k=1}^{n} X_k$. Give a complete proof with all the details that

$$P\left(\lim_{n\to\infty} \frac{S_n - ES_n}{n} = 0\right) = 1$$

[This result is due to Cantelli.]

(b) State, without proof, a stronger version of the result for iid random variables [due to Kolmogorov]. Please keep in mind that you cannot use this result in part (a).