Applied Probability (505A) Graduate Exam

Fall 2011

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. True or false: if A and B are events such that $0 < P(A) < 1$ and $P(B|A) = P(B|A^c)$, then A and B are independent. Justify your answer.

2. Suppose X and Y are independent, each with exponential density e^{-x} for $x > 0$, and let $Z = X - Y$.

(a) Calculate the density of Z.

(b) Calculate the moment generating function of X, the characteristic function of X, and the characteristic function of Z.

3. On the first day of class, the professor observed that there are m men and n women in the class. He says "I will bet even money, that some man-woman pair of students in this class have the same birthday."

Let W denote the number of man-woman pairs which have a birthday in common, so that the professor is betting that $W > 0$. (For example, if Fred, Bob, Mary, Jane, and Linda all have the same birthday, then $W \geq 6$.) You may assume that the birthdays of the $m + n$ students are distributed independently and uniformly over the 365 days of a non-leap year.

(a) Find an exact simple expression for the expectation of W.

(b) Find and simplify the variance of W.

(c) Suppose $m = 10$ and $n = 20$. Name a simple distribution that gives a good approximation for the distribution of W. (There may be more than one acceptable answer; no proof is required.)

(d) With $m = 10$ and $n = 20$, does the professor expect to win or lose money? You can give a heuristic approximation, but your answer should involve some calculation. Recall that $\ln 2 \approx 0.693$.