MATH 505a **QUALIFYING EXAM** **September 21, 2010**

Solve all four problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. Let $X_k,\ k \geq 1$, be iid random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2}$$

exists in an appropriate sense, and identify the limit.

Problem 2. Fix $p \in (0,1)$ and consider independent Poisson random variables $X_k,\ k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}.$$

Verify that the sum $\sum\limits_{k=1}^{\infty} kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum\limits_{k=1}^{\infty} kX_k$.

Suggestion: compute the generating function for X_k, for kX_k, and for Y.

Problem 3. A coin-making machine produces quarters in such a way that, for each coin, the probability U to turn up heads is uniform on $[0,1]$. A coin pops out of the machine.

(a) Compute the conditional distribution of U given that the coin is flipped once and lands on head.

(b) Compute, either exactly or approximately, the conditional distribution of U given that the coin is flipped 2000 times and lands on head 1500 times.

Problem 4. An ordered vertical stack of n books is on my desk. Every day, I pick one book uniformly at random from the stack and put the book on top of the stack. What is the expected number of days before the books are back to the original order?

Comments: (a) For partial credit, just guess the answer, as a function of n. (b) For more credit, give a heuristic justification. (c) For bonus credit, give a proof.