MATH 505a **QUALIFYING EXAM** **February 2, 2011**

Solve all problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, **and write on only one side of the paper**. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. `This is a basic computational problem.`

Let X be normal with zero mean and variance σ^2, let Θ be uniform on $(0, \pi)$, and let a be a real number. Assume X and Θ are independent. Find the density of $Z = X + a\cos(\Theta)$.

Problem 2. `This is an abstract version of the coupon collector problem.`

(a) Suppose that r balls are placed at random into n boxes. Show that if

$$\lim_{n,r\to\infty} ne^{-r/n} = \lambda \in (0, \infty),$$

then, in the same limit, the number of empty boxes has Poisson distribution with mean λ.

(b) Let $X_1, X_2, \ldots$ be iid uniform on $\{1, 2, \ldots, n\}$: for each $k \geq 1$ and $m = 1, \ldots, n$, $P(X_k = m) = 1/n$. Define

$$T_n = \inf\{m : \{X_1, \ldots, X_m\} = \{1, 2, \ldots, n\}\}.$$

That is, T_n is the first time the sample $X_1, \ldots, X_m$ contains all the numbers from 1 to n. Show that

$$\lim_{n\to\infty} P\big(T_n - n\ln(n) \leq nx\big) = \exp(-e^{-x}).$$

Suggestion. Use part (a) and note that $T_n \leq m$ if and only if m balls fill up all n boxes.

Problem 3. `This problem tests your ability to work with indicator random variables and to use the variance-covariance expansion.`

Each member of a group of n players rolls a die (an ordinary fair die with faces number 1 to 6.) For every pair of players who throw the same number, the group scores 1 point. For example, if $n = 10$ and the ordered results of the rolls are 1,2,3,3,3,5,5,5,5,6, then group's score is 3+6=9. In the unlikely event that everyone rolls the same number, the maximal possible score, $\binom{n}{2}$, is achieved.

Find the mean and variance of the total score of the group.