# Numerical Analysis Screening Exam, Spring 2010

**First Name:**

**Last Name:**

## I. Linear Equations

Consider the singular system

$$Bu = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} u = f.$$

(a) Find the range $\mathcal{R}(\mathcal{B})$ and null space $\mathcal{N}(\mathcal{B})$ of $B$.

(b) State the solvability condition for $Bu = f$.

(c) Find an example of $f$ for which $Bu = f$ has *no* solutions.

(d) Find an example of $f$ for which $Bu = f$ has *infinitely many* solutions, and find the explicit form of this solution (depending on a parameter $c$).

## II. Least Squares Problem

Consider the matrix $A$ and vector $b$ given below

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -2 \\ 1 & 5 & 6 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) Using the Householder transformation matrix method to find a $QR$-decomposition of the matrix $A$ and a solution to the problem

$$\min \|Ax - b\|_2^2.$$

(b) Find the minimum norm solution of the above least squares minimization problem.

## III. Eigenvalue Problems

(a) First consider the following matrix

$$B = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}.$$

Find any real eigenvalue of $B$ and any associated eigenvector.

(b) Now let $A$ be any $n \times n$ matrix. Show that $\det(A) = \prod_{i=1}^{n} \lambda_i$ where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $A$. (Hint: Consider characteristic polynomial of $A$)

(c) Show that $A$ is singular if and only if $\lambda = 0$ is an eigenvalue of $A$.

## IV. Iterative Methods

(a) Consider the iterative method:

$$x_{k+1} = Bx_k + c, \quad k = 1, 2, \cdots,$$

where $B$ is a $n \times n$ matrix and $x_0$ and $c$ are arbitrary vectors of $\mathbb{R}^n$. Define the spectral radius $\rho(B)$ of $B$ and show that $x_k$ converges for all initial vectors $x_0$ if $\rho(B) < 1$.

(b) Consider the Richardson iterative scheme

$$x_{k+1} = x_k + \omega(b - Ax_k), \quad k = 1, 2, \cdots,$$

where $A$ is a $n \times n$ matrix and $\omega$ is a positive number. We assume that all the eigenvalues $\lambda_i$ of $A$ are real and satisfy $0 < \alpha \leq \lambda_i \leq \beta$ for $i = 1, \cdots, n$ and for some positive values $\alpha$ and $\beta$. Find the condition on the number $\omega$ in terms of $\alpha$ and $\beta$ such that for any initial vector $x_0$, the sequence $x_k$ converges to the solution of $Ax = b$.

(c) Assuming that $\alpha = \min_{1\leq i\leq n} \lambda_i$ and $\beta = \max_{1\leq i\leq n} \lambda_i$ in (b), what value of $\omega$ leads to the fastest convergence of the scheme.