Numerical Analysis Screening Exam, Fall 2010

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

SIGNATURE:

PROBLEM 1 Let A be an $m \times n$ matrix and b an $m \times 1$ vector. Consider the least squares problem (LS): find x that minimizes $\|Ax - b\|_2^2$.

(a) Give necessary and sufficient conditions for x to be a solution to (LS).

(b) When is this solution unique?

(c) When is the corresponding residual vector $r = Ax - b$ unique?

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the solution x to (LS) that also minimizes $\|x\|_2^2$.

Name:

Name:

PROBLEM 2. A symmetric $n \times n$ matrix A is called Symmetric Positive Definite (SPD) iff $(Ax, x) > 0$ for all $x \neq 0$.

(a) Assume $n = 2$. Show that if A is SPD then A admits a Cholesky factorization, i.e $A = L \cdot L^T$ where L is a nonsingular lower triangular matrix.

(b) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

Show that A is SPD and calculate the Cholesky factorization of A.

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PROBLEM 3.

(a) Why does any eigenvalue solver have to be iterative?

(b) Present the QR-algorithm to solve eigenvalue problems in some detail and state the corresponding theorem for convergence.

(c) Let $A_k = \begin{pmatrix} c & s \\ s & 0 \end{pmatrix}$, where $c = \cos\theta$ and $s = \sin\theta$. Compute A_{k+1} using QR-iteration, and show that the off-diagonal elements of A_{k+1} are smaller than those in A_k.

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PROBLEM 4. Consider a matrix A given by

$$A = \begin{pmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{pmatrix}$$

(a) Verify that $x = [2, 2, 1]^T$ is a solution of $Ax = [1.001, 1.001, 0.001]^T$. Consider the vectors $y = [2.2, 2.2, 1]^T$ and $z = [202, 202, 201]^T$. Verify that $Ax - Ay$ and $Ax - Az$ have the same infinity vector norm.

(b) Find an estimation of the condition number of A using the results of (a).

(c) Verify that

$$A^{-1} = \begin{pmatrix} -899 & 900 & 999 \\ -999 & 1000 & 999 \\ -900 & 900 & 1000 \end{pmatrix}.$$

Determine whether or not

$$B = \begin{pmatrix} -900 & 900 & 1000 \\ -1000 & 1000 & 1000 \\ -900 & 900 & 1000 \end{pmatrix}$$

is a sufficiently good approximation of the matrix A^{-1} to be used in an iterative method of the form $x_{k+1} = x_k - BAx_k + Bb$ for finding the solution of $Ax = b$.

(Hint: $B \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$)

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