Numerical Analysis Screening Exam, Fall 2009

First Name:

Last Name:

PROBLEM 1 (LEAST SQUARES)

1. Define the Least Squares process using QR factorization for solving $Ax = b$, where $A \in \mathbb{R}^{m \times n}$.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

   Using Householder QR factorization solve $Ax = b$.

PROBLEM 2 (EIGENVALUE PROBLEMS)

1. State the Schur Theorem.

2. Using the Gram-Schmidt process define the othonormal set $\{q_1, q_2\}$ for set of vectors $\{v_1, v_2\}$.

3. Find the Schur decomposition of matrix $A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$.

PROBLEM 3 (ITERATIVE METHODS)
Consider an iterative scheme of the form

$$(rI + H)x_{k+1} = (rI - H)x_k + b,$$

where H is a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$ and r is a positive constant.

(a) Rewrite the iteration in the form $x_{k+1} = Bx_k + c$.

(b) Show that the sequence x_k converges for any x_0.

(c) If the matrix is only non-negative definite, does the sequence still converge for any b and x_0?

Problem 4 (Linear Systems)
(1) Let A be any $n \times n$ matrix and $\|\cdot\|$ be any norm on $\mathbb{R}^n$ (Euclidean n-dimensional space). If $\|I - A\| < 1$, then show that A is invertible and derive the estimate

$$\|A^{-1}\| < \frac{1}{1 - \|I - A\|}.$$

(2) An $n \times n$ matrix $A = [a_{i,j}]$ is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{j=1}^{n} |a_{i,j}|, \text{ for } i = 1, \cdots, n.$$

Show that any strictly diagonally dominant matrix A is invertible. (Hint: recall that $\|A\|_\infty = \max_{1 \leq 1 \leq n} \left\{ \sum_{j=1}^{n} |a_{i,j}| \right\}$ and write $A = DB$ where D is the diagonal part of A and show that $\|I - B\|_\infty < 1$.)