Numerical Analysis Screening Exam, Spring 2011

First Name:

Last name:

Student ID Number:

Signature:

Problem 1. (Linear Equations)

(a) Give a definition of matrix A being positive definite.

(b)-i State **any** theorem for solving $Ax = b$ with symmetric positive definite (SPD) matrices.

(b)-ii What are the computational advantages for solving a problem $Ax = b$ with SPD matrices. Be specific.

(c) Find the largest interval for α so that A is positive definite:

$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}.$$

(Hint: use Gaussian elimination.)

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PROBLEM 2. (EIGENVALUE PROBLEMS)

(a) Let $A = [a_{i,j}]$ be an $n \times n$ matrix. Prove Gerschgorin's Theorem which states the following: For $i = 1, \cdots, n$ let $R_i = \sum_{j=1, j\neq i}^{n} |a_{i,j}|$. Every eigenvalue of A falls within one of the closed discs in the complex plane with center at $a_{i,i}$ and radius R_i. (Hint. Let

$$Ax = \lambda x, \tag{1}$$

and assume the largest component of x in absolute value is x_k. Consider the k-equation of (1).)

(b) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that A is positive definite. (Hint. Use Gerschgorin's Theorem to show that A is positive semi-definite. Then consider the equation $Ax = 0$, where the first component of x equals 1.)

(c) Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the eigenvalues of A and verify that Gerschgorin's Theorem holds.

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PROBLEM 3. (ITERATIVE METHODS)

(a) Let L be a strictly lower triangular and U be a strictly upper triangular $n \times n$ matrices. For any positive number ω define

$$B_\omega = (1 - \omega L)^{-1} \left[(1 - \omega) I + \omega U \right].$$

Show that $\rho(B_\omega) \geq |\omega - 1|$ where $\rho(B_\omega)$ is the spectral radius of B_ω. (Hint. Calculate $\det(B_\omega)$ directly and by the product of eigenvalues and compare.)

(b) Consider solving $Ax = b$ where A is an $n \times n$ matrix and $x, b \in \mathbb{R}^n$.

(i) Give the matrix form of the SOR iterative method.

(ii) Using the result in question (a) above, show that the SOR method with parameter ω can only converge for $0 < \omega < 2$.

(c) Consider the equation $Ax = b$, where $x \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Can the SOR method be applied to this equation? Justify your answer.

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PROBLEM 4. (LEAST SQUARES PROBLEM)

(a) Let $A \in R^{m\times n}$. Give a detailed description of the SVD of A and present briefly an algorithm to solve an overdetermined system using SVD.

(b) What are the advantages and disadvantages of using SVD for overdetermined systems.

(c) Consider the overdetermined system $Au = b$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Find the family of least squares solutions using **any** method that is easy for hand calculation, and finally find the minimum-norm solution to the least squares problems.