Spring 2014 Math 541b Exam

1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed samples from the Normal distribution $\mathcal{N}(\theta, \sigma^2)$ having mean θ and variance σ^2.

   (a) Does a Uniformly Most Powerful, or UMP, level α test of $H_0 : \sigma^2 \leq 1$ versus $H_1 : \sigma^2 > 1$ exist if the mean θ is known? If so, find the form of the rejection region of the UMP test, and if not, explain why not.

   (b) Does a UMP level α test of $H_0 : \sigma^2 \leq 1$ versus $H_1 : \sigma^2 > 1$ exist, if both θ and σ^2 are unknown, with the restriction $\theta/\sigma^2 = 2$?

2. Consider a vector $\mathbf{X} = (X_1, X_2, X_3)$ of counts with distribution given by the multinomial distribution with probabilities

$$P(\mathbf{X} = \mathbf{x}) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i}$$

   for $\mathbf{x} = (x_1, x_2, x_3)$, a vector of non-negative integers summing to n, and

$$(p_1, p_2, p_3) = \left(\frac{1}{3} + \frac{\theta}{3}, \frac{2\theta}{3}, \frac{2}{3} - \theta \right) \quad \text{for some } \theta \in (0, 1).$$

   (a) Write out the equation that would need to be solved in order to obtain the maximum likelihood estimate of θ.

   (b) Show that if additional 'missing data' is now introduced to form a 'full model' that a simpler equation then that in part (a) results, and solve it explicitly. Hint: Consider the first cell.

   (c) Specify the steps of an EM algorithm that takes advantage of the simplification obtained by treating the situation as a missing data problem as in part (b).