Spring 2014 Math 541a Exam

1. For known values $x_{i,1}, x_{i,2}, i = 1, \ldots, n$ let

$$Z_i = \beta_1 x_{i,1} + \epsilon_i$$

and

$$Y_i = \beta_1 x_{i,1} + \beta_2 x_{i,2} + \epsilon_i \quad i = 1, \ldots, n,$$

where $\epsilon_i, i = 1, 2, \ldots, n$ are independent normal random variables with mean 0 and variance 1.

(a) Given the data $\mathbf{Z} = (Z_1, \ldots, Z_n)$ compute the maximum likelihood estimate of β_1 and show that it achieves the Cramer-Rao lower bound. Throughout this part and the following, make explicit any non-degeneracy assumptions that may need to be made.

(b) Based on $\mathbf{Y} = (Y_1, \ldots, Y_n)$, compute the Cramer-Rao lower bound for the estimation of (β_1, β_2), and in particular compute a variance lower bound for the estimation of β_1 in the presence of the unknown β_2.

(c) Compare the variance lower bound in (a), which is the same as the one for the model for Y_i where β_2 is known to be equal to zero, to the one in (b), where β_2 is unknown, and show the latter one is always at least as large as the former.

2. Suppose we observe the pair (X, Y) where X has a Poisson(λ) distribution and Y has a Bernoulli($\lambda/(1+\lambda)$) distribution, that is,

$$P_\lambda(X = j) = \frac{\lambda^j e^{-\lambda}}{j!}, j = 0, 1, 2, \ldots$$

and

$$P_\lambda(Y = 1) = \frac{\lambda}{1+\lambda} = 1 - P_\lambda(Y = 0),$$

with X and Y independent, and $\lambda \in (0, \infty)$ unknown.

(a) Find a one-dimensional sufficient statistic for λ based on (X, Y).

(b) Is there a UMVUE (uniform minimum variance unbiased estimator) of λ? If so, find it.

(c) Is there a UMVUE of $\lambda/(1+\lambda)$? If so, find it.