Geometry and Topology Graduate Exam
Spring 2014

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Problem 1. Let X_n denotes the complement of n distinct points in the plane $\mathbb{R}^2$. Does there exist a covering map $X_2 \to X_1$? Explain.

Problem 2. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point z_0 in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let X be the space obtained from the union of D and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

Problem 3. Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the n–dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f\colon B^n \to \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin 0 is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \to B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.)

Problem 4. Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial t}.$$

Problem 5. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 6. Calculate the integral $\int_{S^2} \omega$ where S^2 is the standard unit sphere in $\mathbb{R}^3$ and where ω is the restriction of the differential 2–form

$$(x^2 + y^2 + z^2)(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

Problem 7. Let M be a compact m–dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.