REAL ANALYSIS GRADUATE EXAM

Spring 2014

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, and that $\{f_n\}_{n\geq 1}$ and f are measurable functions on X such that $f_n \to f$ almost everywhere.

(i) Suppose that $\int f^2 \, d\mu < \infty$. Show that f is integrable.

(ii) Suppose that there exists $C < \infty$ such that $\int f_n^2 \, d\mu \leq C$ for all $n \geq 1$. Show that $f_n \to f$ in L^1.

(iii) Give an example where $\int |f_n| \, d\mu \leq 1$ for all $n \geq 1$ but $f_n \not\to f$ in L^1.

2. For what non-negative integer n and positive real c does the integral

$$\int_1^\infty \ln\left(1 + \frac{(\sin x)^n}{x^c}\right) dx$$

(a) exist as a (finite) Lebesgue integral?

(b) converge as an improper Riemann integral?

3. Suppose f is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)| dx = 0.$$

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces such that $\mu(X) > 0$ and $\nu(Y) > 0$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively) such that

$$f(x) = g(y) \quad \mu \times \nu \text{ -almost everywhere on } X \times Y$$

Show that there exists a constant λ such that $f(x) = \lambda$ for μ-a.e. x and $g(y) = \lambda$ for ν-a.e. y.