ALGEBRA QUALIFYING EXAM SPRING 2014

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$. *Hand in solutions in order of the problem numbers.*

1. Let L be a Galois extension of a field F with $\mathrm{Gal}(L/F) \cong D_{10}$, the dihedral group of order 10. How many subfields $F \subseteq M \subseteq L$ are there, what are their dimensions over F, and how many are Galois over F?

2. Up to isomorphism, using direct and semi-direct products, describe the possible structures of a group of order $5{\cdot}11{\cdot}61$.

3. Let I be a nonzero ideal of $R = \boldsymbol{C}[x_1, \ldots, x_n]$. Show that R/I is a finite dimensional algebra over $\boldsymbol{C}$ if and only if I is contained in only finitely many maximal ideals of R.

4. Let R be a commutative ring with 1, and M a noetherian R module. For N a noetherian R module show that $M \otimes_R N$ is a noetherian R module. When N is an artinian R module show that $M \otimes_R N$ is an artinian R module.

5. For $n \geq 5$ show that the symmetric group S_n cannot have a subgroup H with $3 \leq [S_n : H] < n$ ($[S_n : H]$ is the index of H in S_n).

6. Let R be the group algebra $\boldsymbol{C}[S_3]$. How many nonisomorphic, irreducible, left modules does R have and why?

7. Let each of $g_1(x), g_2(x), \ldots, g_n(x) \in \boldsymbol{Q}[x]$ be irreducible of degree four and let L be a splitting field over $\boldsymbol{Q}$ for $\{g_1(x), \ldots, g_n(x)\}$. Show there is an extension field M of L that is a radical extension of $\boldsymbol{Q}$.