Numerical Analysis Preliminary Examination
Monday February 3, 2014

Work all problems and show all your work for full credit. This exam is closed book, closed notes, no calculator or electronic devices of any kind.

1. (a) Let $\{f_k\}_{k=1}^{n}$ be n linearly independent real valued functions in $L_2(a,b)$, and let Q be the $n \times n$ matrix with entries $Q_{i,j} = \int_a^b f_i(x)f_j(x)dx$. Show that Q is positive definite symmetric and therefore invertible.

   (b) Let g be a real valued functions in $L_2(a,b)$ and find the best (in $L_2(a,b)$) approximation to g in span$\{f_k\}_{k=1}^{n}$.

2. Let A be a 3×3 nonsingular matrix which can be reduced to the matrix

$$U = \begin{bmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{bmatrix}$$

   using the following sequence of elementary row operations:

   (i) α_1 times Row 1 is added to Row 2.
   (ii) α_2 times Row 1 is added to Row 3.
   (iii) Row 2 is multiplied by $\frac{1}{\alpha_3}$.
   (iv) α_4 times Row 2 is added to Row 3.

   (a) Find an LU decomposition for the matrix A.

   (b) Let $b = [b_1 \quad b_2 \quad b_3]^T$ be an arbitrary vector in R^3 and let the vector $x = [x_1 \quad x_2 \quad x_3]^T$ in R^3 be the unique solution to the linear system $Ax = b$. Find an expression for x_3 in terms of the $\alpha_i's$, the $b_i's$, and the $u_i's$, $i = 1,2,3$.

3. In this problem we consider the iterative solution of the linear system of equations $Ax = b$ with the following $(n-1) \times (n-1)$ matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

(a) Show that the vectors $x^k = \left(\sin\frac{\pi k}{n}, \quad \sin\frac{2\pi k}{n}, \quad \cdots, \quad \sin\frac{\pi(n-1)k}{n}\right)$, for $k = 1, \cdots, n-1$ are eigenvectors of B_J, the Jacobi iteration matrix corresponding to the matrix A given above.

(b) Determine whether or not the Jacobi's method would converge for all initial conditions x^0.

(c) Let L and U be, respectively, the lower and upper triangular matrices with zero diagonal elements such that $B_J = L + U$, and show that the matrix $\alpha L + \alpha^{-1}U$ has the same eigenvalues as B_J for all $\alpha \neq 0$.

(d) Show that an arbitrary nonzero eigenvalue, λ, of the iteration matrix

$$H(\omega) = (I - \omega L)^{-1}\big((1-\omega)I + \omega U\big)$$

for the Successive Over Relaxation (SOR) method satisfies the following equation

$$\lambda^2 - 2(1-\omega)\lambda - \mu^2\omega^2\lambda + (1-\omega)^2 = 0,$$

where μ is an eigenvalue of B_J (Hint: use the result of (c)).

(e) For $n = 4$, find the spectral radius of $H(1)$.

4. (a) Find the singular value decomposition (SVD) of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

(b) Let $\{\lambda_k\}$ and $\{\sigma_k\}$ be the sets of eigenvalues and singular values of $n \times n$ matrix A. Show that: $\min_k \sigma_k \leq \min_k|\lambda_k|$ and $\max_k \sigma_k \geq \max_k |\lambda_k|$.

(c) Let A be a full column rank $m \times n$ matrix with singular value decomposition $A = U\Sigma V^*$, where V^* indicates the conjugate transpose of V .

(1) Compute the SVD of $A(A^*A)^{-1}A^*$ in terms of U, Σ, and V.

(2) Let $||\cdot|| = sup_{x\neq 0}\frac{|||Ax||_2}{||x||_2}$ be the matrix norm induced by the vector 2-norm, and let σ_{max} be the largest singular value of A. Show that $\left||A|\right| = \sigma_{max}$.