DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2013

1. Show that $u \equiv 0$ is an asymptotically stable solution of
$$u'' + (1 - u^2)u' - (u')^3 + u^5 = 0.$$

2. Consider the matrix
$$\Phi(t) = \begin{bmatrix} t^2+1 & 0 & 0 & 3 \\ 4 & t-1 & 2 & t \\ 1 & 2 & t^3+2 & 2 \\ 3 & 1 & 1 & t \end{bmatrix}.$$
on the interval $t \in (a, b)$.

   (i) Give conditions on a and b so that Φ can be a fundamental matrix for a system
$$x' = A(t)x, \qquad x \in \mathbb{R}^4,$$
   on the entire interval $t \in (a, b)$.

   (ii) Can $(a, b) = (-\infty, \infty)$?

3. Let $f \in C^1(R^n, R^n)$ and assume $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is non-increasing along orbits of
$$x' = f(x), \qquad x(0) = x_0, \tag{1}$$
i.e. if $\phi(t, x_0)$ is the solution of (1) then
$$V(\phi(t_1, x_0)) \leq V(\phi(t_2, x_0)),$$
whenever $t_1 \geq t_2$. Suppose for some x_0, the ω-limit set, $\omega(x_0)$ is nonempty and bounded. Prove the restriction $V|\omega(x_0) =$ constant.

4. Let $U = \{x \in \mathbb{R}^n : |x| > 1\}$, Suppose $u \in C^2(U) \cap C(\bar{U})$ is a bounded solution of the following Dirichlet problem: $\Delta u = 0$ in U and $u = \varphi$ on $\Gamma = \{x \in \mathbb{R}^n : |x| = 1\}$, with $\varphi \in C(\Gamma)$.

   a) If $n = 2$, show that there exists at most one solution of the above problem

   b) If n=3, show that it is possible to have more than one bounded solutions of the above problem. What additional condition should you impose so that the solution is unique?

5. Find the explicit solution $u = u(t, x_1, x_2)$ of
$$\partial_t u + x_2 \partial_{x_1} u - x_1 \partial_{x_2} u = 0, \quad t > 0$$
subject to $u(t = 0, x_1, x_2) = e^{-x_1^2}$.

6. Let $u(x, t)$ solve the wave equation $u_{tt} - c^2 u_{xx} = q(x, t)$ for $x \in \mathbb{R},\ t > 0$. with the initial condition $u(x, 0) = 0$ and $u_t(x, 0) = 0$. Here c is positive and $q(x, t) = (1 - x^2)\sin t$ for $|x| \leq 1$ and $q(x, t) = 0$ for $|x| > 1$. SHow that $u(x, t) = 0$ for $|x| > ct + 1$.

7. Let $u \in W^{1,p}(\mathbb{R}^n)$ where $p > n$. Show that
$$\frac{1}{r^n} \int_{B(x,r)} |u(x) - u(y)|\, dy \leq C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x - y|^n}\, dy$$
where $Du(y)$ is the gradient of u $(Du(y) = (\partial_1 u(y),\ ...., \partial_n u(y))$, $B(x, r)$ is the euclidean ball with center x and radius r, and $C(n)$ is a constant depending only on n.