Fall 2013 Math 541a Exam

1. For $p \in (0,1)$ unknown, let $X_0, X_1, \ldots$ be independent identically distributed random variables taking values in $\{0,1\}$ with distribution

$$P(X_i = 1) = 1 - P(X_i = 0) = p,$$

and suppose that

$$T_n = \sum_{i=0}^{n-1} I(X_i = 1, X_{i+1} = 1). \tag{1}$$

is observed.

   (a) Calculate the mean and variance of T_n.

   (b) Find a consistent method of moments $\widehat{p}_n = g_n(T_n)$ estimator for the unknown p as a function g_n of T_n that may depend on n, and prove that your estimate is consistent for p.

   (c) Show that T_n is not the sum of independent, identically distributed random variables. Nevertheless, determine the non-trivial limiting distribution of $\widehat{p}_n$, after an appropriate centering and scaling, as if (1) was the sum of i.i.d. variables and has the same mean and variance as the one computed in part (a).

   (d) Explain why you would, or would not, expect $\widehat{p}_n$ to have the same limiting distribution as the one determined in part (c).

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density given by

$$f_\beta(x) = \frac{x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} \exp(-x/\beta), \text{ for } x > 0$$

where $\alpha > 0$, and is known. Suppose it is desired to estimate β^3.

   (a) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of β^3.

   (b) Find a complete and sufficient statistic for β. Then, compute its k^{th} moment, where k is an positive integer.

   (c) If a UMVUE (uniform minimum variance unbiased estimator) exists, find its variance and compare it to the bound in part (a).