Geometry/Topology Qualifying Exam

Fall 2013

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. (15 pts) Let X denote S^2 with the north and south poles identified.
   (a) (5 pts) Describe a cell decomposition of X and use it to compute $H_i(X)$ for all $i \geq 0$.
   (b) (5 pts) Compute $\pi_1(X)$.
   (c) (5 pts) Describe (i.e., draw a picture of) the universal cover of X and all other connected covering spaces of X.

2. (10 pts) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective.

3. (10 pts) Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ is a smooth manifold. Here $M_n(\mathbb{R})$ is the set of $n \times n$ real matrices.

4. (10 pts) Compute the de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

5. (10 pts) Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:
$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \ldots$$

6. (10 pts) A lens space $L(p, q)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ for coprime p, q.
   (a) (5 pts) Compute $\pi_1(L(p, q))$.
   (b) (5 pts) Show that any continuous map $L(p, q) \to T^2$ is null-homotopic.

7. (10 pts) Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?