REAL ANALYSIS GRADUATE EXAM

Fall 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let μ be a finite Borel measure on $\mathbb{R}$, which is absolutely continuous with respect to the Lebesgue measure m. Prove that $x \mapsto \mu(A+x)$ is continuous for every Borel set $A \subseteq \mathbb{R}$.

2. Let f be a Lebesgue integrable function on $\mathbb{R}$, and assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Prove that $g(x) = \sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere and is integrable on $\mathbb{R}$. Also, find an example of a Lebesgue integrable function f on $\mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} f(nx)$ converges almost everywhere but is not integrable.

3. Assume $b > 0$. Show that the Lebesgue integral

$$\int_1^{\infty} x^{-b} e^{\sin x} \sin(2x)\, dx$$

exists if and only if $b > 1$.

4. Suppose that F is the distribution function of a Borel measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Prove that

$$\int_{-\infty}^{\infty} \big(F(x+a) - F(x)\big)\, dx = a$$

for all $a > 0$.