Algebra Qualifying Exam - Fall 2013

1. Let H be a subgroup of the symmetric group S_5. Can the order of H be 15, 20 or 30?

2. Let R be a PID and M a finitely generated torsion module of R. Show that M is a cyclic R-module if and only if for any prime $\mathfrak{p}$ of R either $\mathfrak{p}M = M$ or $M/\mathfrak{p}M$ is a cyclic R-module.

3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and suppose I is a proper non-zero ideal of R. The coefficients of a matrix $A \in M_n(R)$ are polynomials in $x_1, \ldots, x_n$ and can be evaluated at $\beta \in \mathbb{C}^n$; write $A(\beta) \in M_n(\mathbb{C})$ for the matrix so obtained. If for some $A \in M_n(R)$ and all $\alpha \in Var(I)$, $A(\alpha) = 0_{n\times n}$, show that for some integer m, $A^m \in M_n(I)$.

4. If R is a noetherian unital ring, show that the power series ring $R[[x]]$ is also a noetherian unital ring.

5. Let p be a prime. Prove that $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. What is the Galois group? (Hint: observe that if α is a root of $f(x)$, then so is $\alpha + i$ for $i \in \mathbb{Z}/p\mathbb{Z}$.)

6. Let R be a finite ring with no nilpotent elements. Show that R is a direct product of fields.

7. Let $K \subset \mathbb{C}$ be the field obtained by adjoining all roots of unity in $\mathbb{C}$ to $\mathbb{Q}$. Suppose $p_1 < p_2$ are primes, $a \in \mathbb{C} \setminus K$, and write L for a splitting field of

$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$

over K. Assuming each factor of $g(x)$ is irreducible, determine the order and the structure of $Gal(L/K)$.