MATH 507a QUALIFYING EXAM September 23, 2013. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. (a) Let $X_1, X_2, \ldots$ be a sequence of random variables such that $|X_n| \le 1$ for all n and $\lim_{n\to\infty} X_n = X$ in probability for some random variable X. Is it true that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = X$$

in probability? Explain your reasoning.

(b) Would the conclusion be different, if the hypothesis $|X_n| \le 1$ for all n were eliminated? If your conclusion was true for (a), and false for (b), you should provide an explicit counterexample.

2. (a) Show that for a nonnegative r.v. X,

$$\sum_{n=1}^{\infty} \mathbf{P}(X \ge n) \le \mathbf{E}X \le 1 + \sum_{n=1}^{\infty} \mathbf{P}(X \ge n).$$

(b) Let X_n be a sequence of i.i.d. r.v.'s with $\mathbf{E}|X_n| = \infty$ for each n.
Show that for $k \ge 1$,

$$\sum_n \mathbf{P}(|X_n| \ge kn) = \infty,$$

and

$$\limsup_n |X_n| n^{-1} = \infty \text{ a.s.}$$

Deduce from here that

$$\limsup_n \left| \frac{X_1 + \ldots + X_n}{n} \right| = \infty \text{ a.s.}$$

3. Let X_n be a sequence of i.i.d. r.v.'s with $\mathbf{E}X_k = \mu > 0$ and $Var(X_k) = \sigma^2$. Let Y_n be a sequence of i.i.d. r.v.'s with $\mathbf{P}(Y_k = 1) = \mathbf{P}(Y_k = -1) = 1/2$. In addition assume (X_n) and (Y_n) are independent. Calculate the limiting distribution of

$$\frac{\sqrt{n}\sum_{k=1}^{n} X_k Y_k}{\sum_{k=1}^{n} X_k}$$

as $n \to \infty$. Specify the distribution fully, including name and parameters, if applicable.