Preliminary Exam in Numerical Analysis Fall 2013

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

For $x \in R^n$ and $M \in R^{n\times n}$ let $\|x\|$ denote the norm of x, and let $\|M\|$ denote the corresponding induced matrix norm of M. Let $S \in R^{n\times n}$ be nonsingular and define a new norm on R^n by $\|x\|_S = \|Sx\|$.

(a) Show that $\| \quad \|_S$ is in fact a norm on R^n.

(b) Show that $\| \quad \|_S$ and $\| \quad \|$ are equivalent norms on R^n.

(c) Show that the induced norm of $M \in R^{n\times n}$ with respect to the $\| \quad \|_S$ norm is given by $\|M\|_S = \|SMS^{-1}\|$.

(d) Let $\kappa(M)$ denote the condition number of $M \in R^{n\times n}$ with respect to the $\| \quad \|$ norm, let $\kappa_S(M)$ denote the condition number of $M \in R^{n\times n}$ with respect to the $\| \quad \|_S$ norm and show that $\kappa_S(M) \le \kappa(S)^2\kappa(M)$.

2. **Least squares**

(a) Assume you observe four (x, y) data points: (0,1), (1, 1), (-1, -1), (2,0). You want to fit a parabola of the form $y = a + bx^2$ to these data points that is best in the least squares sense. Derive the normal equations for this problem and put them in matrix vector form (you do not need to solve the equations).

(b) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and consider the linear system $Ax = b$, for $b \in R^3$. Find the QR or SVD decomposition of A and the rank of A.

(c) For a given $b \in R^3$, state the condition such that the equation in part (b) has a solution, and the condition such that the solution is unique.

(d) Find the pseudoinverse of the matrix A given in part (b).

(e) For $b = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$ find the solution x to the system given in part (b).

3. **Iterative Methods**

Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \qquad (1)$$

where $M \in C^{n\times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) Let $r(M)$ denote the spectral radius of the matrix M and show that if $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then $r(M) < 1$.

Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(b) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Gauss-Seidel iteration for solving the linear system given in (2).

(c) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Successive Over Relaxation Method (SOR) with parameter θ for solving the linear system given in (2). (Hint: Use your answer in part (b) and write Das $D = \frac{1}{\theta}D + \left(1 - \frac{1}{\theta}\right)D$.)

(d) Show that if for the SOR method, $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then it is necessary that $\theta \in (0,2)$.

4. Computation of Eigenvalues and Eigenvectors

Let A be a nondefective $n \times n$ matrix with eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_n$, with $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$, and corresponding eigenvectors $u_1, u_2, \dots, u_n$. Let $x_0 \in C^n$ be such that $x_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_1 \neq 0$. Define the sequence of vectors $\{x_k\}_{k=1}^{\infty} \subseteq C^n$ recursively by $x_{k+1} = Ax_k$, $k = 0,1,2,\dots$

(a) Let $v \in C^n$ be any fixed vector that is not orthogonal to u_1. Show that $q_k = {v^T x_{k+1}}/{v^T x_k}$ converges to $\lambda_1$as $k \to \infty$.

(b) Now suppose that $|\lambda_2| > |\lambda_3|$, $v \in C^n$ is orthogonal to $u_1$but is not orthogonal to u_2 and $\alpha_2 \neq 0$. Show that $\lim_{k\to\infty} q_k = \lambda_2$.

(c) Now suppose $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|$, $v \in C^n$ is such that $\alpha_1 v^T u_1 \neq 0$. Show that for k sufficiently large, $q_k \approx \lambda_1 + C(\lambda_2/\lambda_1)^k$ for some constant C. (Hint: Show that $\lim_{k\to\infty}(q_k - \lambda_1)(\lambda_1/\lambda_2)^k = C$, for some constant C.)