1. (a) Let $f_\theta(x)$, $\theta \in \Theta \subseteq \mathbb{R}$, be a family of density functions with respect to some common measure. If we say that this family has the monotone likelihood ratio (MLR) property in the real-valued statistic $T = T(x)$, what two properties must hold?

   (b) Taking $\Theta = (0, \infty)$, let $f_\theta(x)$, $x = (x_1, \dots, x_n)$, be the joint density of a random sample of n i.i.d. uniform $(0, \theta)$ observations:

$$f_\theta(x) = \begin{cases} \theta^{-n}, & \text{if } x_i < \theta \text{ for all } i = 1, \dots, n \\ 0, & \text{otherwise.} \end{cases}$$

   Show that this family has the MLR property, and give the statistic T.

   (c) Given $\alpha \in (0, 1)$ and $\theta_0 > 0$, find a uniformly most powerful level-α test of

$$H_0 : \theta \le \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0$$

   in terms of $T(X)$. Find any critical values and randomization constants explicitly.

2. Recall that a log-normal distribution $\ln \mathcal{N}(x|\mu, \sigma^2)$ is a continuous probability distribution of a random variable whose logarithm is normally distributed $\mathcal{N}(x|\mu, \sigma^2)$. That is, if $X \sim \ln \mathcal{N}(x|\mu, \sigma^2)$, then $\log X \sim \mathcal{N}(x|\mu, \sigma^2)$. Suppose the only random number generator that you have is the one for log-normal distributions $\ln \mathcal{N}(x|\mu, \sigma^2)$. Propose an MCMC algorithm for estimating the following integral

$$I = \int_0^\infty e^{-x^4 - x^6 - x^8} \frac{e^x}{\alpha} dx,$$

   where $\alpha = \int_0^\infty e^{-x^4 - x^6 - x^8} dx$ (is unknown). Describe the algorithm in detail.