GEOMETRY TOPOLOGY QUALITFYING EXAM
SPRING 2013

Solve all of the problems that you can. Partial credit will be given for partial solutions.

(1) Consider the form

$$\omega = (x^2 + x + y) dy \wedge dz$$

on $\mathbb{R}^3$. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i \colon S^2 \to \mathbb{R}^3$ the inclusion.
  (a) Calculate $\int_{S^2} \omega$.
  (b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

(2) Find all points in $\mathbb{R}^3$ in a neighborhood in which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.

(3) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension n.

(4) (a) Show that every closed 1-form on S^n, $n > 1$ is exact.
  (b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$ is exact.

(5) Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

(6) Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.
  (a) Compute the homology groups $H_*(X, \mathbb{Z})$.
  (b) Compute the fundamental group of X.

(7) (a) Find all of the 2-sheeted covering spaces of $S^1 \times S^1$.
  (b) Show that if a path-connected, locally path connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

(8) (a) Show that if $f \colon S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.
  (b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

Date: February 1, 2013.