

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^{\infty} \frac{x^{1/3}}{1+x^4} dx$$

being careful to justify your answer.

2. Assume that f is an entire function such that

$$|f(z)| \geq \frac{1}{1+|z|} \text{ for all } z \in \mathbb{C}.$$

Prove that f is a constant function.

3. Let $f_n, n \geq 1$, be a sequence of holomorphic functions on an open connected set D such that $|f_n(z)| \leq 1$ for all $z \in D, n \geq 1$. Let $A \subseteq D$ be the set of all $z \in D$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in D , then there exists a holomorphic function f on D such that $f_n \rightarrow f$ uniformly on every compact subset of D as $n \rightarrow \infty$.

4. Let $f(z)$ be meromorphic on \mathbb{C} , holomorphic for $\operatorname{Re} z > 0$ and such that $f(z+1) = zf(z)$ in its domain with $f(1) = 1$.

Show that f has the first order poles at $0, -1, -2, \dots$, and find the residues of f at these points.