Algebra Graduate Exam

Spring 2013

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let $p > 2$ be a prime. Describe, up to isomorphism, all groups of order $2p^2$.

2. Let R be a commutative Noetherian ring with 1. Show that every proper ideal of R is the product of finitely many (not necessarily distinct) prime ideals of R.
(Hint: Consider the set of ideals that are not products of finitely many prime ideals. Also, note that if R is not a prime ring then $IJ=(0)$ for some non-zero ideals I and J of R)

3. In the polynomial ring $R = \mathbf{C}[x,y,z]$ show that there is a positive integer m, and polynomials $f, g, h \in R$ such that
$$(x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5)^m = (x - y)^3 f + (y - z)^5 g + (x + y + z - 3)^7 h.$$

4. Let $R \neq (0)$ be a finite ring such that for any $x \in R$ there is $y \in R$ with $xyx = x$. Show that R contains an identity element and that, for $a, b \in R$, if $ab=1$ then $ba=1$.

5. Let $f(x) = x^{15} - 2$, and let L be the splitting field of $f(x)$ over $\mathbf{Q}$.
a) What is $[L:\mathbf{Q}]$?
b) Show there exists a subfield F of degree 8 that is Galois over $\mathbf{Q}$.
c) What is $\mathrm{Gal}(F/\mathbf{Q})$?
d) Show there is a subgroup of $\mathrm{Gal}(L/\mathbf{Q})$ that is isomorphic to $\mathrm{Gal}(F/\mathbf{Q})$.

6. Let $F/\mathbf{Q}$ be a Galois extension of degree 60, and suppose F contains a primitive ninth root of unity. Show $\mathrm{Gal}(F/\mathbf{Q})$ is solvable.

7. Let n be a positive integer. Show that $f(x,y) = x^n + y^n + 1$ is irreducible in $\mathbf{C}[x,y]$.