Math 507a 2013 Spring Qualifying Exam

1. For each $n = 1, 2, \ldots$, let $X_{1,n}, \ldots, X_{n,n}$ be independent mean zero random variables with variances $\sigma_{i,n}^2 = \mathrm{Var}(X_{i,n})$ satisfying $\sum_{i=1}^n \sigma_{i,n}^2 = 1$. Recall that the Lindeberg-Feller Central limit theorem says that the sum $W_n = \sum_{i=1}^n X_{i,n}$ converges in distribution to the standard normal $\mathcal{N}(0,1)$ whenever the Lindeberg condition holds, that is, when

$$\lim_{n\to\infty} L_{n,\epsilon} = 0 \quad \text{for all } \epsilon > 0, \quad \text{where} \quad L_{n,\epsilon} = \sum_{i=1}^n E\left(X_{i,n}^2 \mathbf{1}(|X_{i,n}| \ge \epsilon)\right).$$

a. Prove that the Lindeberg condition is satisfied when $X_{1,n} = X_1/\sqrt{n}, \ldots, X_{n,n} = X_n/\sqrt{n}$ and $X_i, i = 1, 2 \ldots$, are independent and identically distributed.

b. Prove that the Lindeberg condition is satisfied when

$$\lim_{n\to\infty} \sum_{i=1}^n E|X_{i,n}|^p = 0 \quad \text{for some } p > 2.$$

c. Prove that the Lindeberg condition is not necessary for the weak convergence of W_n to $\mathcal{N}(0,1)$.

2. Let X be a random variable such that $m_n = EX^n$ exists for all $n = 1, 2, \ldots$.

a. Prove that H_n, the $n \times n$ matrix with entries m_{i+j-2}, is non-negative definite.

b. Prove that if $P(X \in [0,1]) = 1$, then

$$(-1)^k \Delta^k m_n \ge 0 \quad \text{for all } n, k \ge 0,$$

where $\Delta m_n = m_{n+1} - m_n$.

3. Let $Y, Y_1, Y_2, \ldots$ be independent and identically distributed, with the unit exponential distribution, $P(Y > t) = e^{-t}$ for positive real t. Let

$$W = \gamma + \sum_{k \ge 1} \frac{Y_k - 1}{k},$$

where $\gamma = \lim_{n\to\infty}\left((1+\frac{1}{2}+\cdots+\frac{1}{n}) - \log n\right)$ is Euler's constant.

a) Compute the characteristic function of W, $\phi(u) := Ee^{iuW}$.

b) Show that $\int_{-\infty}^{\infty} |\phi(u)|\, du < \infty$.

c) Does the finiteness property in b) prove that W has an absolutely continuous distribution?

d) Find the density $f(w)$ of W. Even if you cannot simplify the integral, be sure to show the appropriate Fourier inversion formula. Hint: Consider $W_n = \gamma + \sum_{1\leq k\leq n} \frac{Y_k - 1}{k}$ and relate W_n to the maximum of n independent exponentially distributed random variables.