Math 505a 2013 Spring Qualifying Exam

1. a) Let X and Y be square integrable random variables such that

$$E(X|Y) = Y \quad \text{and} \quad E(Y|X) = X. \tag{1}$$

Show that

$$P(X = Y) = 1. \tag{2}$$

b) Prove that (1) implies (2) under the weakened assumption that X and Y are integrable.

2. Suppose k balls are tossed into n boxes, with all n^k possibilities equally likely. Let D be the number of boxes that contain exactly 2 balls.

a) Compute $p := P($ exactly 2 balls land in box 1$)$.

b) In terms of p, give an exact expression for the mean ED.

c) Compute $r := P($ exactly 2 balls land in box 1 and exactly 2 balls land in box 2$)$.

d) Give an exact expression for the second moment ED^2 in terms of p and r.

e) Compute the variance of D.

3. a) Suppose $g(u) := Eu^S$ is the probability generating function of a nonnegative integer valued random variable S satisfying $P(S > 0) > 0$. Let T be distributed as S, conditional on the event $S > 0$. Express $h(u) := Eu^T$, the probability generating function of T, in terms of $g(u)$.

In parts b) and c) below, N is a nonnegative integer valued random variable with probability generating function $f(u) := Eu^N$, and S is the number of heads in N tosses of a $p \in (0, 1)$ coin, with all coin tosses having probability p of coming up heads, independently of each other and of N.

b) Write the probability generating function $g(u) := Eu^S$ of S in a simple form.

c) Now combine parts a) and b): what is the probability generating function h of the number T of heads, in N tosses of a p-coin, conditional on getting at least one head, when N has probability generating function f?
Parts d,e) can be worked on even if you are stumped by a,b,c).

d) Suppose someone claims that for $\alpha \in (0,1)$, the function

$$f(u) := 1 - (1-u)^\alpha$$

is a probability generating function of a nonnegative, non constant integer valued random variable N. What properties of f must you check? Is the hypothesis $\alpha > 0$ used? What happens in the cases $\alpha = 0, \alpha = 1$ and $\alpha > 1$?

e) Combine parts a)-d), that is suppose $\alpha \in (0,1)$, N has the generating function $f(u) := 1 - (1-u)^\alpha$, and T is the number of heads in N tosses of a p-coin, conditional on getting at least one head. Do N and T have the same distribution?