Numerical Analysis Screening Exam, Spring 2013

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. At the end of the first step of Gaussian Elimination without partial pivoting, we have:

$$A_1 = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline & & & \\ 0 & & & \\ \vdots & & \hat{A} & \\ 0 & & & \end{array}\right)$$

a. Show that $\hat{A}$ is also a SPD.

b. Use the first conclusion to show the existence of the LU factorization and Cholesky factorization of any SPD.

Problem 2.

A matrix A with all non-zero diagonal elements can be written as $A = D_A(I-L-U)$ where D_A is a diagonal matrix with identical diagonal as A and matrices L and U are lower and upper triangular matrices with zero diagonal elements. The matrix A is said to be consistently ordered if the eigenvalues of matrix $\rho L+\rho^{-1}U$ are independent of $\rho \neq 0$. Consider a tri-diagonal matrix A of the form

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \beta & \alpha \end{pmatrix}$$

with $|\alpha| \geq 2\beta > 0$.

a. Show that the matrix A is consistently ordered.

b. Show that if $\lambda \neq 0$ is an eigenvalue of the iteration matrix B_ω of the Successive Over Relaxation (SOR) method for matrix A

$$B_\omega = (I - \omega L)^{-1}((1-\omega)I + \omega U),$$

then $\mu = (\lambda + \omega - 1)(\omega\sqrt{\lambda})^{-1}$ is an eigenvalue of $L + U$.

Problem 3.

a. Assume that $v_1 = (1, 1, 1)^T$ is an eigenvector of a 3×3 matrix B. Find a real unitary matrix V such that the first column of the matrix $V^T BV$ contains all zeros except on the first row.

b. Consider a matrix A defined by

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

verify that $v_1 = (1, 1, 1)^T$ is an eigenvector of A and the first column of the matrix $V^T AV$ contains all zeros except on the first row where V is the matrix you obtained in (a).

c. Assume that $V^T AV$ has the form

$$V^T AV = \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

Find a Schur decomposition of the matrix A. That is, find a unitary matrix U such that $U^H AU = R$ where R is an upper triangular matrix and U^H is the conjugate transpose of U.

Problem 4.

Consider a n by m matrix A and a vector $b \in \mathbb{R}^n$. A minimum norm solution of the least squares problem is a vector $x \in \mathbb{R}^m$ with minimum Euclidian norm that minimizes $\|Ax - b\|$. Consider a vector x^* such that $\|Ax^* - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^n$. Show that x^* is a minimum norm solution if and only if x^* is in the range of A^*.