DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2012

1. Consider the product space $\mathbb{R}^2 \times \mathbb{R}$ and let $P : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x,t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^2$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

where $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = A(t)y, \quad y(t_1) = x$, evaluated at t_2, and

$$A(t) = \begin{pmatrix} -2 + \sin t & \cos t \\ \sin 4t & 1 + \cos t \end{pmatrix}.$$

Show (1) T is a diffeomorphism and (2) if μ is Borel measure then

$$\mu(T(B)) \to 0 \quad \text{as} \quad t_2 \to 0,$$

for any Borel set B.

2. Define the mapping $f : \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$ by

$$f(x) = \frac{ax^2}{1 + x^2}, \quad a > 0,\ x \geq 0.$$

As a increases, starting near zero, a bifurcation takes place. Describe the type of bifurcation, find the critical value a_{c} and the point x_{bif} at which the bifurcation takes place.

3. Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} x + \begin{pmatrix} x_1^2 \\ 2x_2^2 \end{pmatrix}.$$

Show

(i) The first quadrant $x_1 \geq 0,\ x_2 \geq 0$, call it Q_+, is invariant, i.e. $f(Q_+) \subset Q_+$ and

(ii) $x \in Q_+,\ x \neq (0,0), \implies \|f^n(x)\| \to \infty$ as $n \to \infty$, where

$$f^n \doteq f \circ f \circ \cdots \circ f.$$

4. Let u be a smooth solution of

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty)$$
$$u = g, u_t = h \text{ on } \mathbb{R}^3 \times \{t = 0\},$$

where g and h are smooth and have compact support. Prove the existence of $C > 0$ such that

$$|u(x,t)| \leq \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

5. (a) Solve the linear partial differential equation

$$e^x u_x + u_y = u \text{ with } u(x,0) = g(x).$$

(b) Solve the nonlinear partial differential equation

$$x^2 u_x + y^2 u_y = u^2 \text{ with } u = 1 \text{ on the line } y = 2x$$

6. Write down the explicit formula for the solution of

$$\begin{aligned} u_t - \Delta u + cu &= f \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$