Geometry-Topology Qualitfying exam
Fall 2012

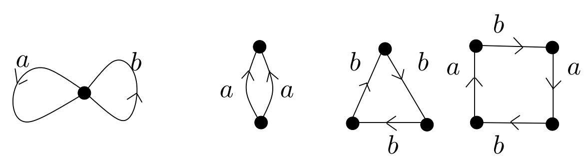
Solve all of the problems. Partial credit will be given for partial answers.

1. Denote by $S^1 \subset \mathbf{R}^2$ the unit circle and consider the torus $T^2 = S^1 \times S^1$. Now, define $A \subset T^2 = S^1 \times S^1$ by
$$A = \{(x, y, z, w) \in T^2 \mid (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\ \}.$$
Compute $H^*(T^2, A)$. *Here we regard S^1 as a subset of the plane, hence we indicate points on S^1 as ordered pairs.*

2. Denote by S^1 and S^2 the circle and sphere respectively. Recall that the definition of the smash product $X \wedge Y$ of two pointed spaces is the quotient of $X \times Y$ by $(x, y_0) \sim (x_0, y)$.

   Show that $S^1 \times S^1$ and $S^1 \wedge S^1 \wedge S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

3. Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.



1-skeleton 2-skeleton

   (a) Compute the homology of X.
   (b) Present the fundamental group of X and prove its nonabelian.

   (*Justify your work.*)

4. Does there exist a smooth embedding of the projective plane $\mathbf{R}P^2$ into $\mathbf{R}^2$? Justify your answer.

5. Let M be a manifold, and let $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbf{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider the vector fields X and Y as maps $C^\infty(M) \to C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$? Explain.

6. Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbf{R}^4$. Compute $\int_S \omega$ where $\omega = (w + w^2) dx \wedge dy \wedge dz$.

7. Does the equation $x^2 = y^3$ define a smooth submanifold in $\mathbf{R}^3$? Prove your claim.