REAL ANALYSIS GRADUATE EXAM

Fall 2012

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m be the Lebesgue measure on $X = [0,1]$. If

$$m(\limsup_{n\to\infty} A_n) = 1, m(\liminf_{n\to\infty} B_n) = 1,$$

prove that $m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1$, where

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \liminf_{n\to\infty} B_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k.$$

2. Assume $f : X \to [0,\infty)$ is measurable. Find

$$\lim_n \int_X n \log\left[1 + \frac{f(x)}{n}\right] d\mu.$$

3. Let $f \in L^1(m)$. For $k = 1, 2, \ldots$ let f_k be the step function defined by

$$\begin{aligned} f_k(x) &= k\int_{j/k}^{(j+1)/k} f(t)dt \\ \text{for } \frac{j}{k} &< x \le \frac{j+1}{k}, j = 0, \pm 1, \ldots. \end{aligned}$$

Show that f_k converges to f in L^1 as $k \to \infty$.

4.If E is Borel set in $\mathbb{R}^n$ the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))},$$

whenever the limit exists [Here m denotes the Lebesgue measure and $B(x,r)$ is the open ball with center at x and radius r.]

(a) Show that $D_E(x) = 0$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.
(b) For $\alpha \in (0,1)$ find an example of E and x such that $D_E(x) = \alpha$.
(c) Find an example of E and x such that $D_E(x)$ does not exist.