Algebra Graduate Exam

Fall 2012

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

2. Let A, B, and C be finitely generated $F[x] = R$ modules, for F a field, with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

3. Working in the polynomial ring $\boldsymbol{C}[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

4. For integers n, $m > 1$, let $A \subseteq M_n(\boldsymbol{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that A is commutative. Is the converse true?

5. Let F be the splitting field of $f(x) = x^6 - 2$ over $\mathbf{Q}$. Show that $\mathrm{Gal}(F/\mathbf{Q})$ is isomorphic to the dihedral group of order 12.

6. Given that all groups of order 12 are solvable show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.