Math 507a 2012 Fall Qualifying Exam

1. Let $X_n, n \geq 1$ be random variables such that X_n has probability density function

$$f_n(x) = \begin{cases} 1 + \sin(2\pi n x) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the sequence $\{X_n,\ n \geq 1\}$ converges in distribution as $n \to \infty$ and identify the limit.

(b) Assume in addition that X_n are independent. Find a sequence a_n of real numbers such that with probability one,

$$0 < \limsup_{n\to\infty} (\log X_n)/a_n < \infty.$$

If you are unable to do b) as stated, replace X_n with U_n, i.i.d. variables with the uniform distribution on $[0, 1]$.

2. A random variable X is called infinitely divisible if, for every $n = 1, 2, 3, \ldots$ there exist independent and identically distributed random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ is that of X.

a. Suppose that X_m is a sequence of infinitely divisible random variables with $\sup_m \mathrm{Var}(X_m) < \infty$, and that X_m converges to X in distribution. Prove that X is infinitely divisible. Hint: Use tightness.

b. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} 1 - |x| & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show X is not infinitely divisible. Hint: Consider that X has bounded support.

c. Let U be a random variable uniformly distributed over the interval $[-1/2, 1/2]$. Show U is not infinitely divisible. (You may use part b.)

3. Suppose μ is a nonnegative, sigma-finite measure on $(0, \infty)$ such that

$$m := \int_0^\infty x\, \mu(dx) \in (0, \infty).$$

Let $i = \sqrt{-1}$ and let ϕ be given by

$$\phi(u) := \exp\left(\int (e^{iux} - 1)\, \mu(dx) \right).$$

a) Prove that there is a random variable T such that $Ee^{iuT} = \phi(u)$ for all real u. Hint: With δ_x point mass at $x > 0$ and $\lambda > 0$, consider the special cases $\mu = \lambda \delta_1$, $\mu = \lambda \delta_x$, then $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$.

b) Assuming a), prove $m = ET$.

c) Assuming a), give a simple integral expression for the variance of T.