Math 505a 2012 Fall Qualifying Exam

1. In the Polya Urn model, $w \geq 1$ white balls and $b \geq 1$ black balls are placed in an urn at time 0, and at times $1, 2, \ldots$ a ball is chosen uniformly from the urn independent of the past, and replaced back into the urn with one additional ball of the same color.

a. A vector $(X_1, \ldots, X_n)$ of random variables is said to be exchangeable

$$(X_1, \ldots, X_n) =_d (X_{\pi(1)}, \ldots, X_{\pi(n)}) \quad \text{for all permutations } \pi$$

where $=_d$ denotes equality in distribution. If X_i is the indicator that a white ball is drawn from the urn at time i, prove that $(X_1, \ldots, X_n)$ is exchangeable.

b. Find the mean and variance of $S_n = X_1 + \cdots + X_n$, the total number of white balls added to the urn up to time n.

2. With a and b positive numbers, a needle of length $l \in (0, \min(a, b)]$ is dropped randomly on a rectangular grid consisting of an infinite number of parallel lines distance a apart, and, perpendicular to these, an infinite number of parallel lines distance b apart. Let A and B, respectively, be the events that the needle intersects the group of lines at distance a and b apart.

a. Show $P(A) = \frac{2l}{a\pi}$ and $P(B) = \frac{2l}{b\pi}$. Hint: The angle θ giving the orientation of the needle might be taken as uniform from $[0, 2\pi)$, but by symmetry, one may assume that the angle is uniformly taken from $[0, \pi/2)$.

b. Determine $P(A \cap B)$ and verify that A and B are strictly negatively correlated, that is, that $P(A \cap B) < P(A)P(B)$.

3. A total of k boys and $n - k$ girls sit around a circular table, with all $n!$ arrangements equally likely. Compute the mean and variance of the number Y of pairs of boy/girl neighbors. Note: In the circular arrangement GBGGBB, since the first G and last B are neighbors, $Y = 4$.