Numerical Analysis Screening Exam, Fall 2012

Direct Methods for Linear Equations.

a. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. There exists a nonsingular lower triangle matrix L satisfying $A = L \cdot L^t$. Is this factorization unique? If not, propose a condition on L to make the factorization unique.

b. Compute the above factorization for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 13 & 8 \\ 1 & 8 & 14 \end{pmatrix}.$$

Iterative Methods for Linear Equations.

Consider the iterative method:

$$Nx_{k+1} = Px_k + b, k = 0, 1, \cdots,$$

where N, P are $n \times n$ matrices with $detN \neq 0$; and x_0, b are arbitraray $n-$dim vectors. Then the above iterates satisfy the system of equations

$$x_{k+1} = Mx_k + N^{-1}b, k = 0, 1, \cdots \tag{1}$$

where $M = N^{-1}P$. Now define $N_\alpha = (1+\alpha)N, P_\alpha = P + \alpha N$ for some real $\alpha \neq -1$ and consider the related iterative method

$$x_{k+1} = M_\alpha x_k + N_\alpha^{-1}b, \quad k = 0, 1, \cdots, \tag{2}$$

where $M_\alpha = N_\alpha^{-1}P_\alpha$.

a. Let the eigenvalues of M be denoted by: $\lambda_1, \lambda_2, \cdots, \lambda_n$. Show that the eigenvalues $\mu_{\alpha,k}$ of M_α are given by:

$$\mu_{\alpha,k} = \frac{\lambda_k + \alpha}{1+\alpha}, \quad k = 1, 2, \cdots, n$$

b. Assume the eigenvalues of M are real and satisfy: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 1$. Show that the iterations in eq. (2) converge as $k \to \infty$ for any α such that $\alpha > \frac{1+\lambda_1}{2} > -1$.

Eigenvalue Problem.

a. Let λ be an eigenvalue of a $n \times n$ matrix A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$ for any polynomial $f(x) = \sum_{k=0}^n a_k x^k$.

b. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric matrix satisfying:

$$a_{1i} \neq 0, \qquad \sum_{j=1}^n a_{ij} = 0, \qquad a_{ii} = \sum_{j\neq i} |a_{ij}|, \qquad i = 1, \cdots, n$$

Show all eigenvalues of A are non-negative and determine the dimension of eigenspace corresponding to the smallest eigenvalue of A.

Least Square Problem.

a. Let A be an $m \times n$ real matrix with the following singular value decomposition: $A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T & V_2^T \end{pmatrix}^T$, where $U = (U_1 \ U_2)$ and $V = (V_1 \ V_2)$ are orthogonal matrices, U_1 and V_1 have $r = rank(A)$ columns, and Σ is invertible.

For any vector $b \in \mathbb{R}^n$, show that the minimum norm, least squares problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

always has a unique solution, which can be written as $x = V_1 \Sigma^{-1} U_1^T b$.

b. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using part a) above, find the minimum norm, least squares solution to the problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

Hint: You can assume that the U in the SVD of A must be of the form $U = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ for some real $a > 0$.