1. A sample of size n is drawn without replacement from an urn containing N balls, m of which are red and $N-m$ are black; the balls are otherwise indistinguishable. Let X denote the number of red balls in the sample of size n. In what follows we treat N, n as known and m as unknown.

   (a) Find $P_m(X = x)$.

   (b) Show that
   $$\widehat{m} = \min\{\lfloor X(N+1)/n \rfloor, N\}. \tag{1}$$
   is an MLE of m.

   (c) Define
   $$\underline{x}_{m,\alpha} = \max\{x \in \mathbb{Z} : P_m(X \le x) \le \alpha\}$$
   $$\overline{x}_{m,\alpha} = \min\{x \in \mathbb{Z} : P_m(X \le x) \ge \alpha\}.$$
   Show that
   $$\{m : \underline{x}_{m,\alpha/2} < X \le \overline{x}_{m,1-\alpha/2}\} \tag{2}$$
   is a $100(1-\alpha)\%$ confidence interval for m, possibly conservative. *Hint:* Invert a hypothesis test of $H_0 : m = m_0$ vs. $H_1 : m \neq m_0$, and note that one of the inequalities in (2) is strict.

2. (a) Let $S_1 \sim \text{Bin}(n_1, p)$ and $S_2 \sim \text{Bin}(n_2, p)$ be two independent binomial random variables, and let $S = S_1 + S_2$. Identify the distribution of S_1 conditional on $S = s$, and give its parameter values in terms of an urn model.

   (b) Now let $S_1 \sim \text{Bin}(n_1, p_1)$ and $S_2 \sim \text{Bin}(n_2, p_2)$ be independent, and $S = S_1 + S_2$ as above. Fisher's Exact Test of $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$ rejects H_0 when S_1 is large.

      i. Show that, under H_0, S is a sufficient statistic.

      ii. Write down an expression for the p-value of Fisher's Exact Test, conditional on $S = s$, in terms of the density $f(s_1|s)$ of the distribution in part 2a.