1. (a) Let Z_i be independent $N(0,1), i = 1, 2, \cdots, n$. Are $\overline{Z} = \frac{1}{n}\sum_{i=1}^n Z_i$ and $S_Z^2 = \frac{1}{n-1}\sum_{i=1}^n (Z_i - \overline{Z})^2$ independent? Prove your claim.

   (b) Let $X_1, X_2, \cdots, X_n$ be independent identically distributed normal with mean θ and variance θ^2, where $\theta > 0$ is unknown. Let

   $$\overline{X} = \frac{\sum_{i=1}^n X_i}{n}, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2.$$

   Are $\overline{X}$ and S^2 independent? Prove your claim. (Hint: you can directly use the result in the first part of this problem.)

   (c) Show that $(\overline{X}, S^2)$ is a sufficient statistic for θ, but it is not complete.

2. (a) Let $X_1, X_2, \cdots, X_n$ be exponentially distributed with density

   $$f(x) = \lambda \exp(-\lambda x), \quad x > 0.$$

   Let $c > 0$ be a constant and if $X_i < c$, we observe X_i, otherwise we observe c.

   $$S_n = \sum_{i=1}^n X_i I(X_i < c), \quad T_n = \sum_{i=1}^n I(X_i > c),$$

   where $I(A) = 1$ if event A occurs and $I(A) = 0$, otherwise. Write down the likelihood function of the observed values in terms of T_n and S_n.

   (b) Show the maximum likelihood estimator of λ is

   $$\widehat{\lambda}_n = \frac{n - T_n}{S_n + cT_n}.$$