Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

2. (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{100}\}$.
   (a) Compute the homology of $\Sigma_{1,1}$.
   (b) Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2.

3. (10 pts) Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that
$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$
where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

4. (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1, 1] \times [-1, 1])/((1, y) \sim (-1, -y))$.
   (a) Determine the fundamental group of X.
   (b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

5. (10 pts) Determine all the connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

6. (10 pts) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

7. (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:
$$X(x_1, x_2, x_3, x_4) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$
$$Y(x_1, x_2, x_3, x_4) = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$
Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.