Spring 2012 Real Analysis Exam

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f and g be real integrable functions on a σ-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let

$$F_t = \{x \in E : f(x) > t\} \quad \text{and} \quad G_t = \{x \in E : g(x) > t\}.$$

Show that

$$\int_X |f - g| d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \backslash G_t) \cup (G_t \backslash F_t)\right) dt.$$

2. Show that

$$\int_{\pi}^{\infty} \frac{dx}{x^2 (\sin^2 x)^{1/3}}$$

is finite.

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M \to \infty} \sup_{\alpha \in \mathcal{A}} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0.$$

a. Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

b. Prove that if $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{f_\beta\}_{\beta \in \mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma \in A \cup B}$ is uniformly integrable.

c. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^{\infty} \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^{\infty}$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

4. Let $\mathbb{M}$ be the collection of all finite measures on the measure space $(X, \mathcal{M})$.

a. Show that

$$d(\nu,\lambda) = 2 \sup_{E\in\mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

b. For any $\mu \in \mathbb{M}$ that dominates measures ν and λ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu,\lambda) = \int |p(x) - q(x)|\, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\})\, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.