Spring 2012 Complex Analysis Exam

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a > 0$. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\,dx,$$
being careful to justify your methods.

2. Let $f(z)$ be analytic for $0 < |z| < 1$. Assume there are $C > 0$ and $m \geq 1$ such that
$$|f^{(m)}(z)| \leq \frac{C}{|z|^m}, 0 < |z| < 1.$$
Show that f has a removable singularity at $z = 0$.

3. Let $D \subseteq \mathbf{C}$ be a connected open subset and let (u_n) be a sequence of harmonic functions $u_n : D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.

4. Let D be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, and define $\Omega = D \setminus [0, 1]$. Find a conformal mapping of Ω onto D. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.