ALGEBRA QUALIFYING EXAM SPRING 2012

Work all of the problems. *Justify the statements in your solutions by reference to specific results*, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\boldsymbol{Q}$ and set of the complex numbers is $\boldsymbol{C}$.

1. Let I be an ideal of $R = \boldsymbol{C}[x_1, \ldots, x_n]$. Show that $\dim_C R/I$ is finite $\Leftrightarrow$ I is contained in only finitely many maximal ideals of R.

2. If G is a group with $|G| = 7^2 \cdot 11^2 \cdot 19$, show that G must be abelian and describe the possible structures of G.

3. Let F be a finite field and G a finite group with GCD$\{char\ F, |G|\} = 1$. The group algebra $F[G]$ is an algebra over F with G as an F-basis, elements $\alpha = \sum_G a_g g$ for $a_g \in F$, and multiplication that extends $ag \cdot bh = ab \cdot gh$. Show that any $x \in F[G]$ that is not a zero left divisor (i.e. if $xy = 0$ for $y \in F[G]$ then $y = 0$) must be invertible in $F[G]$.

4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \boldsymbol{Q}[x]$ and if $\boldsymbol{Q} \subseteq M \subseteq \boldsymbol{C}$ is a splitting field for $p(x)$ over $\boldsymbol{Q}$, argue that Gal$(M/\boldsymbol{Q})$ is solvable.

5. Let R be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1y_1 + \cdots + x_ny_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \ldots, r_n) \in R^n \mid x_1r_1 + \cdots + x_nr_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that A has n generators, and that when $R = F[x]$ for F a field then A_R is free of rank $n - 1$.

6. For p a prime let F_p be the field of p elements and K an extension field of F_p of dimension 72.
   i) Describe the possible structures of $Gal(K/F_p)$.
   ii) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that K is a splitting field of $g(x)$ over F_p.
   iii) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree d that split in K?