Preliminary/Qualifying Exam in Numerical Analysis (Math 502a) Spring 2012

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems

(a) What is the LU-decomposition of an n by n matrix A, and how is it related to Gaussian elimination? Does it always exist? If not, give sufficient condition for its existence.

(b) What is the relation of Cholesky factorization to Gaussian elimination? Give an example of a symmetric matrix for which Cholesky factorization does not exist.

(c) Let $\mathrm{C} = \mathrm{A} + \mathrm{iB}$ where A and B are real n by n matrices. Give necessary and sufficient conditions on A and B for C to be Hermitian, and give a nontrivial example of a 3 by 3 Hermitian matrix.

2. Least squares

(a) Give a simple example which shows that loss of information can occur in forming the normal equations. Discuss how the accuracy can be improved by using iterative improvement.

(b) Compute the pseudoinverse, $x^\dagger$, of a nonzero row or column vector, x, of length n. Let a = [1, 0] and let b = $[1, 1]^T$. Show that $(ab)^\dagger \neq b^\dagger a^\dagger$.

3. Iterative Methods

Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \qquad (1)$$

where $M \in C^{n\times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) If $x^* \in C^n$ is a fixed point of (1) and $\|M\| < 1$ where $\|\cdot\|$ is any compatible matrix norm induced by a vector norm, show that x^* is unique and that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$.

(b) Let $r(M)$ denote the spectral radius of the matrix M and use the fact that $r(M) = inf\|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$ if and only if $r(M) < 1$.

Now consider the linear system

$$Ax = b \qquad (2)$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(c) What are the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2).

(d) Use part (a) to show that if $A \in C^{n\times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

(e) Use part (b) together with the Gershgorin Circle Theorem to show that if $A \in C^{n\times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

4. Computation of Eigenvalues and Eigenvectors

Consider an $n \times n$ Hermitian matrix A and a unit vector q_1. For $k = 2, \cdots n$, let $p_k = Aq_{k-1}$ and set

$$q_k = \frac{h_k}{\|h_k\|_2}, \qquad h_k = p_k - \sum_{j=1}^{k-1} (q_j^H \cdot p_k)\, q_j.$$

where $\|\cdot\|_2$ is the Euclidian norm in C^n.

(a) Show that the vectors q_k, for $k = 1, \cdots n,$ form an orthogonal set if none of the vectors h_k is the zero vector.

(b) Consider the matrix $Q^H AQ$. Use part (a) to show that it is a tridiagonal matrix (Hint: $[Q^H AQ]_{i,j} = q_i^H Aq_j$).

(c) Suggest a possible approach that uses the result of part (b) to reduce the number of operations in the QR-algorithm for the computation of the eigenvalues of the matrix *A*.