Topics for the Graduate Exam in Complex Analysis

Most of the following topics are normally covered in the course Math 520.

This is a one hour exam.

Elementary properties of holomorphic functions: Power series representation, integral representation (Cauchy's theorem for "nice" domains). Cauchy-Riemann equations. Taylor series, Cauchy integral formula, classification of isolated singularities, meromorphic functions. Liouville's theorem. The elementary holomorphic functions (rational functions, the exponential and logarithm functions, trigonometric functions, powers and roots).

The residue theorem and its applications: Evaluating integrals by the methods of residues, counting zeros and poles. Rouche's theorem, open mapping theorem, inverse and implicit function theorems. Methods for computing residues.

Harmonic functions: Mean value property and maximum principle for harmonic and analytic functions. Realization of a real harmonic function as the real part of an analytic function (construction of the conjugate harmonic function in a simply connected domain). Poisson integral formula. Schwarz's lemma.

Limits of analytic functions: Properties carried over by uniform convergence of compact subsets, various hypotheses under which one may deduce uniform convergence on compact subsets, normal families.

Conformal mapping: Local mapping properties of analytic functions, the elementary mappings (Mobius transformations, exp(z), log(z), etc.), Riemann mapping theorem.

Analytic continuation: Reflection across analytic boundaries (Schwarz reflection principle), conformal mapping of polygons to the disk, Picard's theorem.

References:

L.V. Ahlfors, Complex Analysis
W. Rudin, Real and Complex Analysis
E. Hille, Analytic Function Theory, Vol. I, II
J.B. Conway, Functions of One Complex Variable

COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1992

Problem 1 Compute the following integrals

(a) $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,\mathrm{d}x$

(b) $\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta}$

Problem 2 Map the region inside the circle $|z| = 1$ and outside the circle $|z - 1/2 = 1/2$ conformally onto the unit disk $\{z : |z| < 1\}$.

Problem 3 Determine all entire $f(z)$ such that $\Re f(z) > 1$ and $\Im f(z) < -1$, where $\Re$ and $\Im$ denote the real and imaginary part.

Problem 4 How many roots does the equation

$$z^{15} - 2z^{11} + 7z^3 - 2z^2 + 1 = 0$$

have in the unit disk $|z| < 1$?

COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1993

Problem 1 Compute the following integral using residues

$$\int_0^\infty \frac{x^{\frac{1}{2}} \log x}{(x^2+1)^2}\, \mathrm{d}x$$

Problem 2 Map the upper half disk $\{z : |z| < 1, \Im z > 0\}$ conformally onto the unit disk $\{z : |z| < 1\}$.

Problem 3 How many roots does the equation

$$z^{11} + 4z^{10} - z^9 + 12z^5 - 2z^4 + z - 1 = 0$$

have in the annulus $1 < |z| < 2$?

Problem 4 Determine all entire functions $f(z)$ such that

$$|f(z)| \leq C|e^z|(|z|^2+1)\log|z|$$

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1993

Problem 1 Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, \ldots$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

Problem 2 Prove or find a counterexample: Let D be a countable dense subset of $(0,1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0,1)$.

Problem 3 Let f be a non-constant meromorphic function which is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

Problem 4 How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2$$

Hint: consider the image of the imaginary axis.

Problem 5 Show that a function $f : (a,b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

Problem 6 Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Problem 7 Suppose f is a non-negative function which is Lebesgue integrable on $[0,1]$, and $\{r_n : n = 1, 2, \ldots\}$ is an enumeration of the rational numbers in $[0,1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$$

converges for a.e. $x \in [0,1]$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1994

Problem 1 Evaluate $\int_0^\infty \frac{\log x}{1+x^2}\, dx$

Problem 2 Show that $[0,1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

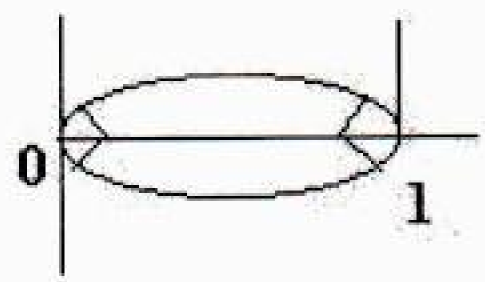
Problem 3 Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

$$|f(z)| \quad |f(0)|\frac{1+|z|}{1 \quad |z|}$$

Problem 4 Let $f : [1, +\infty) \to [0, +\infty)$ be Lebesgue measurable. Prove:

$$\int_1^\infty \frac{f(x)^2}{x^2} < +\infty \quad \Rightarrow \quad \int_1^\infty \frac{f(x)}{x^2}\, dx < +\infty$$

Problem 5 Map the region between the circular arcs (in the figure below) conformally to the unit disk. (Note that the top and bottom acrs intersect at right angles.)



Problem 6 Let $([0,1], \mathcal{A}, \;)$ denote the Lebesgue measure space on $[0,1]$. Give examples to show that for $f : [0,1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is imlied by, the condition "there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e."

Problem 7 An entire function is said to have **finite order** if there exists $c > 0$ such that $|f(z)| \quad \exp(|z|^c)$ for all $|z|$ sufficiently large; the **order** of f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $1/2$:

$$f(z) = \prod_{k=1}^{\infty} 1 \quad \frac{z}{k^2}$$

Problem 8 Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \;)$ with $\;(X) < \infty$. We say the sequence is **uniformly integrable** if

$$\lim_{R\to\infty} \sum_n \int_{|f_n| \quad R} |f_n|\, d \quad = 0$$

(a) Show that if there exists $g \in L^1(\;)$ such that $|f_n(x)| \quad g(x)$ for all x, n there the $\{f_n\}$ are uniformly integrable.

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\;)$ and

$$\lim_n \int f_n\, d \quad = \int f\, d$$

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1994

Problem 1 If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, must the derivative f' be continuous at SOME point of $\mathbb{R}$? Explain. (Write f' as the limit of a sequence of continuous functions.)

Problem 2 Evaluate

$$\int_{f(\Gamma_\rho)} \frac{\mathrm{d}z}{1+z}$$

where Γ_ρ is the square with verticies $\pm\rho$, $\pm i\rho$ and $f(z) = e^z$.

Problem 3 Let $(X, \mathcal{B}, v)$ be a finite measure space and f_n, f nonnegative bounded measurable functions on X. Define measures μ_n, μ by

$$\mu_n(A) = \int_A f_n \,\mathrm{d}v, \quad \mu(A) = \int_A f \,\mathrm{d}v$$

Prove: $f_n \to f$ in L^1 iff $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$ as $n \to \infty$.

Problem 4 Evaluate $\int_0^\infty \frac{x^{1/3}}{4+x^4}\,\mathrm{d}x$

Problem 5 Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ which converges in the L^2 sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the L^2 sense. Prove: $\{f_n\}$ converges to 0 uniformly.

Problem 6 Conformally map the open unit disk to a semi-infinite strip in the plane, $\{z \in \mathbb{C} : \Re z > 0, 0 < \Im z < a\}$ for some $a > 0$.

Problem 7 Suppose $(X, \mathcal{B}, \mu)$ is a finite measure space and f_n, g_n are measurable real-valued functions on X such that $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Show that given $\varepsilon > 0$ there exists an M such that $\mu(\{x \in X : |f_n(x)| > M\}) < \varepsilon$ for all n.

(b) Show that $f_n g_n \to fg$ in measure.

Spring 1995

Analysis Qualifying Examination
Tuesday, May 2, 1995

INSTRUCTIONS. Do **any seven** of the following problems; begin each problem on a fresh sheet of paper.

Problem 1. *Let $f(z)$ be analytic in $|z| \le 1$, with $f(0) = a_0 \neq 0$. If $M = \max_{|z|=1} |f(z)|$, show that $f(z) \neq 0$ for*

$$|z| < \frac{|a_0|}{|a_0| + M}.$$

Problem 2. *Let μ be a finite measure on $(X, \mathcal{B})$, suppose $f_n \to f$ a.e. on X, and $\|f_n\|_2 \le M < \infty$ for all n. Show that $f_n \to f$ in L^1.*

Problem 3. *Compute:*

$$\int_{|\xi|=1} \frac{d\xi}{\sqrt{\xi}}.$$

Problem 4. *Suppose μ is a measure on $(X, \mathcal{B})$, and suppose the function $f : \mathbb{R} \times X \to \mathbb{R}$ is differentiable in the L^2 sense, that is, $f(t, \cdot) \in L^2(\mu)$ for all t and $(f(t+h, \cdot) - f(t, \cdot))/h$ converges in $L^2(\mu)$ to a limit $g(t, \cdot)$ as $h \to 0$. Define*

$$\alpha(s,t) = \int_X f(s,x) f(t,x)\, d\mu(x).$$

Prove: $\dfrac{\partial^2 \alpha}{\partial t \partial s}$ exists.

Problem 5. *Show that $f(z) = z^5 + z^3 + 2z + 3$ has only one zero in the first quadrant $x \ge 0, y \ge 0$ (where $z = x + iy$).*

Problem 6. *Investigate the convergence of $\sum_n u_n$, where*

$$u_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

Problem 7. *If $f(z)$ is analytic for $|z| < 1$ and $f(0) = 0$, prove that*

$$\sum_{n=1}^{\infty} f(z^n)$$

converges in $|z| < 1$ to an analytic function.

Problem 8. *If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that its graph*

$$G = \{(x, f(x)) : x \in \mathbb{R}\}$$

has Lebesgue measure zero in $\mathbb{R}^2$. (Hint: first do it for bounded functions.)

Ken

Fall 1995

Graduate Exam in Analysis, Fall 1995

1. We say that "$f_n \to f$ almost in $L^1(\mu)$" if for all $\epsilon > 0$ there exists a set N such that $\mu(N) < \epsilon$ and $\int_{N^c} |f_n - f| d\mu \to 0$ as $n \to \infty$.

a) Show that if f_n converges to f almost in L^1, and f_n converges to g almost in L^1, then $f = g$ a.e.

Consider the following statements:

(1) f_n converges to f in L^1.

(2) f_n converges to f almost in L^1.

(3) f_n converges to f in measure.

b) Show that (1) implies (2) implies (3).

c) Show that neither of the two reverse implications in part b) hold.

2. Let μ be a measure on the Borel subsets of R^n. With τ denoting the collection of open subsets of R^n, we define the support of μ by

$$support(\mu) = \{x \in R^n : \mu(U) > 0 \text{ for all } U \in \tau \text{ with } x \in U\}.$$

Prove that the set $support(\mu)$ is closed.

3. If f_n is a sequence of continuous functions on $[0,1]$ with $f_n \to f$ a.e. m (Lebesgue measure), prove that for any $0 \leq a < 1$, $[0,1]$ contains a compact subset K such that $m(K) > a$ and f is continuous on K. (Hint: Apply Egoroff's theorem.)

X 4. Let I be the collection of bounded open intervals (a,b) of R and m Lebesgue measure. Prove there is no Borel set E such that $m(A \cap E) = \frac{1}{2}m(A)$ for all $A \in I$.

5. Evaluate the following integral: $\int_0^\infty \frac{x \cos x}{x^2+1} dx$.

6. Conformally map the region $\{z = x + iy \in C : y > \frac{1}{4} - x^2\}$ to the unit disk $|z| < 1$.

7. Let $f(z)$ be a bounded analytic function on $|z| < 1$, and $\{z_n\}$ be the zeroes of $f(z)$, is it true that $\sum(1 - |z_n|) < \infty$?

Qualifying Exam, Spring 1996

Complex Analysis: Do 4 of the following problems.

1. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx, \quad \int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}dx.$$

2. If f is analytic on $\{0 < |z| < 1\}$, can e^f have a pole at 0?

3. Suppose f is analytic on $\{|z| < 1\}$ and that $f(0) = f(1/2) = 0$ and $|f|_\infty \leq 1$, find the best pointwise bound for $|f(z)|$ and discuss when the bound is achieved.

4. Find a conformal map of the region D bounded by the two circles $C_1 : \{|z| = 1\}$, $C_2 : \{|z - (1/2)| = 1/2\}$ onto the unit disk.

5. Is the set of analytic functions satisfying the bound $\int_\Delta |f(z)|^2 dxdy \leq 1$ on the unit disc $\Delta = \{|z| < 1\}$ a normal family?

6. Suppose $f(z) = \sum a_n z^n$ with the property that $\lim_{n\to\infty} a_n = a$. Does $f(z)$ have a pole at $z = 1$ with residue $-a$? and is $f(z) - \frac{a}{1-z}$ analytic for $|z| < \rho, \quad \rho > 1$? Provide a proof if yes, a counterexample if no.

7. What can you say about the location of the zeroes of the polynomials $1 + z + \frac{z^2}{2} + ... + \frac{z^n}{n}$ for n sufficiently large?

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1996

Problem 1 (Stability of contractive iteration) Let (M, d) be a metric space, and suppose $T : M \to M$ satisfies

$$d(Tx, Ty) \le k \cdot d(Tx, Ty) \quad \text{for all} \quad x, y \in M$$

where $0 < k < 1$. Now suppose $\varepsilon > 0$, and a sequence $\{\hat{x}_n\}_{n=0}^{\infty}$ in M satisfies

$$d(\hat{x}_n, T\hat{x}_{n-1}) < \varepsilon \quad \text{for all} \quad n \ge 1$$

Prove that for $0 \le m < n$,

$$d(\hat{x}_m, \hat{x}_n) < k^n \frac{2d(\hat{x}_0, T\hat{x}_0)}{1-k} + \frac{2\varepsilon}{1-k}$$

Problem 2 How many zeros does the polynomial $p(z) = z^4 - 2z + 3$ have in the unit disk $|z| < 1$?

Problem 3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty} \varphi(x) f(x)\ \mathrm{d}x = 0$$

for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ which have compact support. Prove: $f(x) = 0$ for a.e. x.

Problem 4 Evaluate

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{2 + \sin\theta}$$

Problem 5 Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let M denote the space of Σ-measurable extended-real-valued functions on X. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|}\ \mathrm{d}\mu$$

Show that ρ is a metric on M, and that $f_n \to f$ in the ρ-metric iff $f_n \to f$ in measure.

Problem 6 Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Prove that there exists a point $z_0 \in \mathbb{C}$ such that we can expand $f(z)$ into a power series about z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for which all $c_n \ne 0$.

Problem 7 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives f_{xy} and f_{yx}. Prove $f_{xy} \equiv f_{yx}$. **Hint:** use Fubini's theorem to integrate f_{xy} and f_{yx} over a rectangle $[a, b] \times [c, d]$.

Problem 8 Find a conformal mapping from the unit disk $|z| < 1$ to the region

$$\Omega = \{x + iy : (x < 0) \text{ and } (y > 0),\ \text{or } (x \ge 0) \text{ and } (y > b)\}$$

where $b > 0$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1997

Directions: Do any **seven** of the following eight problems.

Problem 1 Prove: if $n \geq 2$ is an integer, then

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{x/n}{\sin(\pi/n)}$$

Problem 2 Suppose Ω is an open connected region of the complex plane and f is a non-constant analytic function on $\overline{\Omega}$. Prove: if $|f(z)| \equiv 1$ on the boundary of Ω, then $f(z)$ has at least one zero in Ω.

Problem 3 Formally, we have that

$$\begin{aligned} \frac{(-1)^n n!}{t^{n+1}} &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{t}\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^\infty e^{-tx}\,\mathrm{d}x \\ &= \int_0^\infty \frac{\partial^n}{\partial t^n} e^{-tx}\,\mathrm{d}x = \int_0^\infty (-1)^n x^n e^{-tx}\,\mathrm{d}x \end{aligned}$$

so that on setting $t = 1$ we obtain

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$$

Justify the calculation.

Problem 4 Let $X = C[0,1]$ be the space of all bounded continuous functions from $[0,1]$ to $\mathbb{R}$ with the sup-norm distance,

$$d(f,g) = \sup_{0\leq t\leq 1} |f(t) - g(t)|$$

You may assume that (X,d) is complete. Let $F : X \to X$ be a strict contraction, i.e., a function such that there exists $k < 1$ with

$$d(Fx, Fy) \leq kd(x,y) \text{ for all } x, y \in X$$

Let I denote the identity operator on X, prove:

- $I + F$ is a 1-1 mapping of X onto X
- $(I+F)^{-1}$ is continuous

Problem 5 Let $K : [0,1]\times[0,1] \to \mathbb{R}$ be continuous, and let $\mathcal{F}$ be the family of all functions f on $[0,1]$ of the form

$$f(x) = \int_0^1 g(y)K(x,y)\,\mathrm{d}y$$

Problem 6 Show that for each $\varepsilon > 0$ the function

$$f(z) = \sin z + \frac{1}{z}$$

has infinitely many zeros in the strip $|\Im z| < \varepsilon$.

Problem 7 Determine the order of the entire function

$$f(z) = \prod_{n=1}^\infty \left(1 + \frac{z}{n^2}\right)$$

(Recall that the *order* of an entire function f is

$$\lim_{r\to\infty} \frac{\log\log M(r)}{r}$$

where $M(r) = \max_{|z|=r} |f(z)|$.)

Problem 8 Prove: if A and B are Lebesgue-measurable subsets of $\mathbb{R}$ with positive Lebesgue measure, then the set

$$A + B = \{a + b : a \in A, b \in B\}$$

has non-empty interior. (Hint: consider the convolution of the characteristic functions of A and B.)

Problem 4. Determine all entire functions $f(z)$ for which

$$|f(z)| \leq C|z|^{3/2}$$

for all $|z|$ sufficiently large.

Problem 5. Evaluate:

$$\int_0^{+\infty} \frac{\cos 2ax - 1}{x^2}\, dx \qquad (a \text{ real}).$$

Problem 6. Let $u(z)$ be harmonic in the closed unit disk $\overline{D}$, with

$$u(x) = 0 \text{ for } x \in [-1, 1].$$

Show that $u(z) = -u(\overline{z})$.

SPRING 1998 ANALYSIS QUALIFYING EXAM
MONDAY, MAY 4, 1998

Directions. Do any seven of the following eight problems, **using the paper and pens provided**. Start each problem on a *fresh* sheet of paper. When you have completed the exam, be sure your name is printed on each page; sign the envelope, and return the exam papers in the envelope. You may keep this printed page.

Problem 1. Suppose $f \in L^1(d\mu)$. Prove: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set A with $\mu(A) < \delta$, there holds

$$\left| \int_A f \, d\mu \right| < \varepsilon.$$

Problem 2. Let f be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove: if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

Problem 3. Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0, 1]$ which converges in the L^1 sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the L^1 sense. Prove: $\{f_n\}$ converges to zero uniformly.

Problem 4. Suppose D is the open unit disk in $\mathbb{C}$, and $f : D \to D$ satisfies $f(1/2) = 1/2$. Show that $|f'(1/2)| \leq 3/4$.

Problem 5. Let $(X, \mathcal{T})$ be a topological space which has the property that every closed set F is the intersection of a *countable* family of open sets. Prove: any finite measure μ on the Borel field of $(X, \mathcal{T})$ is *regular*: for each Borel set E and each $\varepsilon > 0$, there exist an open set $G \supset E$ and a closed set $F \subset E$ such that $\mu(G \backslash F) < \varepsilon$. (Hint: consider the collection of Borel sets E for which this condition is true.)

Problem 6. Let D be the open unit disk in $\mathbb{C}$, and let $f : D \to D$ be analytic with $f(0) = 0$. Suppose

$$|f(z)| \geq \frac{1}{6} \quad \text{for all} \quad |z| = \frac{1}{4}.$$

Show that f assumes every value in the disk $|w| < \frac{1}{6}$.

Problem 7. Let $g : [0, 1] \to \mathbb{R}$ be Lebesgue measurable, and suppose $f(x, y) := g(x) - g(y)$ is Lebesgue integrable on $[0, 1] \times [0, 1]$. Prove: g is Lebesgue integrable.

Problem 8. Evaluate: $\int_0^\infty \frac{\sqrt{x}}{1 + x^3} \, dx$.

Q4: ... f analytic ... Show that $|f'(\tfrac{1}{2})| \leq 1$

announced 11:38am

Real and Complex Analysis Exam, Fall 1998

		Problem	Grade
First Name:			
Last Name:		1	
Social Security Number:		2	
		3	
You should answer all 6 questions		4	
Please attach these two pages to your solution pages.		5	
		6	
		Total	

PROBLEM 1. Evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{2x^2+x+1}{x^4+5x^2+4}\,dx \quad \text{and} \quad \int_0^{2\pi} \frac{1}{3+\cos\theta}\,d\theta.$$

PROBLEM 2. In each case either produce a function f analytic in a neighborhood of 0 with the given property or else prove that no such function exists:

(i) $|f^{(n)}(0)| \geq (n!)^2$ for all $n \geq 0$;

(ii) $f(\frac{1}{n}) = (n^2-1)^{-1}$ for all $n \geq 2$;

(iii) $|f(\frac{1}{n})| \leq e^{-n}$ for all $n \geq 1$ and f is not identically zero near 0.

PROBLEM 3. Let D denote the open unit disc $|z| < 1$. For each $a \in D$ let S_a be a fractional linear transformation of D to itself which sends a to 0. Suppose that f is analytic mapping of D to itself such that $f(a_1) = f(a_2) = \cdots = f(a_n) = 0$ for points $a_1, a_2, \ldots a_n \in D$.

(i) Write down a formula for S_a.

(ii) Show that

$$|f(z)| \leq \prod_{r=1}^{n} |S_{a_r}(z)| \qquad \text{for all} \quad z \in D.$$

(iii) Deduce that $|f(0)| \leq |a_1 a_2 \cdots a_n|$.

Analysis Qualifying Exam

Spring, 1999

- In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.
- Start each problem on a fresh sheet of paper.

REAL ANALYSIS. **Do only three of the following four problems.**

1. Suppose f_n, where $n = 1, 2, \ldots$, and f are nonnegative functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Show that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for every measurable E. (Hint: Use Fatou's Lemma.)

2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $E \in \mathcal{M} \otimes \mathcal{N}$ (the product σ-algebra in $X \times Y$). Show that every section $E_x = \{y \in Y : (x, y) \in E\}$ is measurable.

3. Let A denote the set of all $f \in C[0,1]$ such that f is monotonic on some open subinterval of $[0,1]$. Show that A is meager (that is, of the first category) in $C[0,1]$ in the topology of uniform convergence.

4. (a) Show that the class of all step functions, of form $\sum_{j \le n} c_j \chi_{(a_j, b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$ where μ is the Lebesgue measure on $\mathbb{R}$. (Hint: Why is the corresponding statement true for simple functions?)

   (b) Suppose $f \in L^1(\mu)$. Show that $\lim_{h \to 0} \int \left| f(x+h) - f(x) \right| dx = 0$. (Hint: Use (a).)

COMPLEX ANALYSIS. **Do all four problems.**

5. Suppose that f is analytic on $\mathbb{C}$ and that f is a homeomorphism of $\mathbb{C}$ onto a set U.

   (a) Show that f has a non-essential singularity at ∞.

   (b) Deduce that f must be of the form $f(z) = az + b$ for some $a \neq 0$ and that $U = \mathbb{C}$.

6. (a) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant M such that $\left| f^{(k)}(0) \right| \le k^2 M^k$ for all $k \ge 1$. Show that f can be extended to be analytic on $\mathbb{C}$.

   (b) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant $M > 1$ such that $\left| f(1/k) \right| \le M^{-k}$ for $k \ge 2$. Show that f is identically zero.

7. Suppose that f is analytic on $0 < |z| < 2$ and satisfies

$$\int_{|z|=1} z^n f(z)\, dz = 0$$

for $n = 0, 1, 2, \ldots$. Show that f has a removable singularity at 0.

8. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2}\, dx\,.$$

COMPLEX ANALYSIS. Answer any three of the four questions.

5. Starting from Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

(with appropriate conditions on the function f, the curve γ and the point z), explain briefly how to obtain

(a) Cauchy's estimates on $|f^{(n)}(a)|$ in terms of $\sup\{|f(z)| : |z-a| = r\}$;

(b) the result that an analytic function on a connected open set has isolated zeros (unless it is identically zero).

6. (a) Suppose that f is analytic on the set $\{z \in \mathbf{C} : |z| > R\}$, for some R. Define what is means for f to have an essential singularity at ∞. Prove that if f is one-to-one on the set $\{z \in \mathbf{C} : |z| > R\}$ then f has a non-essential singularity at ∞.

(b) Suppose that f is an entire function and has a non-essential singularity at ∞. Prove that f is a polynomial.

(c) Suppose that f is entire and is one-to-one on $\mathbf{C}$. Prove that f is of the form $f(z) = az + b$ where $a \neq 0$.

7. Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

[Hint: consider the function $(1 - e^{2iz})/z^2$.]

8. Suppose that u is harmonic on the open disc $\{z \in \mathbf{C} : |z| < 1\}$ and $u(z) > 0$ for all $z \in D$ and $u(0) = 1$.

(a) Show that u is the real part of some analytic function f on D with $f(0) = 1$. Show also that $|f(z) - 1| < |f(z) + 1|$ for all $z \in D$.

(b) By applying Schwarz lemma to a suitable function, show that $1/3 \leq u(1/2) \leq 3$.

COMPLEX ANALYSIS. Answer any three of the four questions.

5. One of the consequences of Cauchy's integral formula is the result: "if f is analytic on an open ball $B(a,r) = \{z \in \mathbf{C} : |z-a| < r\}$ then either $f \equiv 0$ on $B(a,r)$ or else there exist $n \geq 0$ and an analytic function g on $B(a,r)$ with $g(a) \neq 0$ such that $f(z) = (z-a)^n g(z)$ on $B(a,r)$."

(a) Indicate the main steps in the proof of the result above.

(b) Deduce that if f_1 and f_2 are analytic functions on a connected open set U and if $f_1(z_n) = f_2(z_n)$ for some sequence $z_n \to z_\infty$ with $z_n \in U$ for all n and $z_\infty \in U$ then $f_1 \equiv f_2$.

6. Show that

$$\int_0^\infty \frac{x^a}{1+x^2}\,dx = \frac{\pi}{2\cos(a\pi/2)}$$

for $-1 < a < 1$. You should be careful to justify your calculations. In particular you should show where the assumptions on a are used.

7. Suppose that f is analytic on an open subset U of $\mathbf{C}$ and that f is a one-to-one mapping of U onto a subset V. Show that

(a) $f^{-1} : V \to U$ is continuous;

(b) $f'(z) \neq 0$ for all $z \in U$;

(c) V is open and $f^{-1} : V \to U$ is analytic.

8. Let f be analytic on an open disc $\{z \in \mathbf{C} : |z| < r\}$ for some $r > 1$.

(a) Suppose that $|f(z)| < 1$ if $|z| = 1$. How many fixed points (that is, solutions of $f(z) = z$) does f have in the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$?

(b) Suppose instead that $|f(z)| > 2$ if $|z| = 1$, and $f(0) = 1$. Does f have to have a zero (that is, a solution of $f(z) = 0$) in the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$?

COMPLEX ANALYSIS QUALIFYING EXAM (MATH 520)

FALL 2000

(1) Compute
$$\int_0^\infty \frac{\sin x}{x}\, \mathrm{d}x$$

(2) How many roots does the equation
$$z^6 - 5z^2 + 8z + 2 = 0$$
have in the unit disk $|z| \leq 1$?

(3) Let f be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove that if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

(4) Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds:
if f is holomorphic with $|f| < 1$ in $\{z : |\operatorname{Im} z| < 2\}$, and if $|f(z)| < \delta$ for all $x \in \mathbb{R}$ with $-1 < x < 1$, then $|f| < \epsilon$ in $\{z : |\operatorname{Im} z| < 1\}$.
HINT: First do the problem for $\{z : |z| < 1\}$ in place of $\{z : |\operatorname{Im} z| < 1\}$.

ANALYSIS QUALIFYING EXAM

MAY 10, 2001

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Last Name: ____________________

First Name: ____________________

Social Security Number: ____________________

If you are taking the Real Analysis (525a) exam only: Do problems 1–4.

If you are taking the Complex Analysis (520) exam only: Do problems 5–8.

If you are taking both parts: In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.

Do as many problems as you can.

COMPLEX ANALYSIS PROBLEMS

(5) Compute

$$\int_0^\infty \frac{x^{1/3}\,dx}{(1+x)(4+x)}$$

(6) Find the number of roots of the equation

$$\cos z = ez^3$$

in $\left\{z \in \mathbb{C} : |z| < 1\right\}$ counting multiplicity.

(7) Let n be a positive integer. Find the maximum value of $|f(z)|$ on $\{z \in \mathbb{C} : |z| \leq 1\}$, where

$$f(z) = \left(\frac{1+z}{2}\right)^n - \left(\frac{1+z}{2}\right)^{n-1}.$$

(8) Find all functions f which are holomorphic on $\Pi = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Im} z < \pi/2\}$ such that $f(\Pi) \subseteq \Pi$ with $f(0) = 0$ and $f'(0) = i$.

5. Compute

$$\int_0^\infty \frac{\sin x \, dx}{x(x^2+1)}.$$

6. Denote $D = \{z \in \mathbb{C} : |z| < 1\}$. Assume that a holomorphic function $f : D \to D$ satisfies $f(0) = f'(0) = 0$. Prove that $|f''(0)| \leq 2$ and $|f(z)| \leq |z|^2$ for all $z \in D$.

7. Prove that the function $f(z) = z + 2 - e^z$ has precisely one zero in the half plane $\Pi = \{z \in \mathbb{C} : \Re z < 0\}$.

8. Show that the function $\ln|z|$ has no harmonic conjugate in the region $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Complex Analysis Qualifying Exam
USC Department of Mathematics
September 26, 2002

Instructions. Do all of the following problems, *on separate pieces of paper.* $\mathbb{D}$ denotes the open unit disk of the complex plane.

Problem 1. Suppose f, g are holomorphic in $\mathbb{D}$ and satisfy $|f(z)| = |g(z)|$ for all $z \in \mathbb{D}$. Prove: there exists a constant λ with $|\lambda| = 1$ such that $f = \lambda g$.

Problem 2. How many zeros does

$$f(z) = z^5 + 5z^2 - 1$$

have in the annulus $\{z : 1 < |z| < 2\}$?

Problem 3. Compute

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx.$$

Justify all steps.

Problem 4. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and assume f has two distinct fixed-points (points for which $f(z) = z$). Prove: f is the identity map.

FALL 2003 COMPLEX ANALYSIS (MATH 520) QUALIFYING EXAM
MONDAY, SEPTEMBER 22, 2003

DIRECTIONS. Do **four** of the following five problems. Start each problem on a *fresh* sheet of paper, and write on only one side. When you have completed the exam, be sure your name is printed on each page. You may keep this printed page.

Problem 1. Let f be analytic on an open disc $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 1$.

(a) Suppose that $|f(z)| < 1$ if $|z| = 1$. How many fixed points (that is, solutions of $f(z) = z$) does f have in the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$?

(b) Suppose also that $|f(z)| < 1$ if $|z| = 1$, and $f(1/2) = 1/2$. Describe the set of possible values of $f(0)$. [That is, find a set U so that (i) $f(0) \in U$ for every f satisfying the conditions; and also (ii) for every $a \in U$ there exists an f satisfying the conditions with $f(0) = a$.]

Problem 2. Evaluate the integral

$$\int_0^\infty \frac{x \sin ax}{x^4 + 1}\, dx$$

for $a > 0$, being careful to justify your methods.

Problem 3. Suppose that u is harmonic on the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and $u(z) > 0$ for all $z \in D$ and $u(0) = 1$.

(a) Show that u is the real part of some analytic function f on D with $f(0) = 1$. Show also that $|f(z) - 1| < |f(z) + 1|$ for all $z \in D$.

(b) By applying Schwarz' lemma to a suitable function, show that $1/3 \leq u(1/2) \leq 3$.

Problem 4. Let $f(z)$ be analytic for $z \neq 0$, and suppose that $f(1/z) = f(z)$. Suppose also that $f(z)$ is real for all z on the unit circle $|z| = 1$. Prove that $f(z)$ is real for all real $z \neq 0$.

Problem 5. If $f(z)$ is analytic for $|z| < 1$, and

$$|f(z)| \leq \frac{1}{1 - |z|},$$

show that

$$|a_n| = |f^{(n)}(0)/n!| \leq (n+1)(1 + 1/n)^n < (n+1)e.$$

COMPLEX ANALYSIS QUALIFYING EXAM
SPRING 2004

Answer all four questions. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

1. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{a + \cos x}\, dx$$

for $a > 1$, being careful to justify your methods. What happens if $0 < a < 1$?

2. Prove that there exist distinct points z_1, z_2 in D so that $D \setminus \{z_1, z_2\}$ and $D \setminus \{1/2,\ 1/3\}$ are not conformally equivalent.

3. Prove that there is NO holomorphic function f defined on D with

$$\lim_{|z| \to 1^-} |f(z)| = \infty.$$

4. Let f be a holomorphic function on $\mathbb{C} \setminus A$ for some finite set $A = \{a_1, a_2, \ldots, a_k\}$. Suppose $|f(z)| \to \infty$ as $z \to a_j$ for each $j = 1, \ldots, k$ and that $|f(z)|$ is bounded as $|z| \to \infty$. Show that f is a rational function.

COMPLEX ANALYSIS GRADUATE EXAM

FALL 2004

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

1. Evaluate the integral
$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}\,dx,$$
being careful to justify your methods.

2. Suppose that f is an analytic function on D satisfying $\mathrm{Re} f(z) \geq 0$ for all $z \in D$ and $f(0) = 1$. Prove that
$$|f(z)| \leq \frac{1+|z|}{1-|z|}.$$

3. Prove that $D \setminus [-\frac{1}{2}, \frac{1}{2}]$ and $D \setminus \{0\}$ are not conformally equivalent.

4. Suppose that f is analytic on D except at the point 0 where f has a simple pole with residue n, for some positive integer n. Show that there is an analytic function g on D such that $f = g'/g$ and g has a zero of order n at 0.

COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx = \frac{\pi}{\sin \pi a}$$

if $0 < a < 1$.

2. In each case determine whether or not there is a function f holomorphic on the disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with the given property. If so, identify the function.

(i) $$f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{n} \qquad \text{for } n = 1, 2, \ldots.$$

(ii) $$f\left(\frac{1}{n}\right) = \frac{1}{n+2} \qquad \text{for } n = 1, 2, \ldots.$$

(iii) $$f\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}} \qquad \text{for } n = 1, 2, \ldots.$$

3. Let f be a holomorphic function in $H = \{z \in \mathbb{C} : \mathbb{R}e\, z > 0\}$, and assume that it satisfies

$$f(z+1) = zf(z), \qquad z \in H.$$

Prove that f can be extended to a holomorphic function in

$$\mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$

Prove that if f can be extended to an entire function, then it has an infinite number of zeros.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in a domain $\Omega \subset \mathbb{C}$ which converges to a function f, as $n \to \infty$, uniformly on every compact subset of Ω. Suppose that each function f_n has at most m zeros (counted with multiplicity) in Ω for some fixed $m \in \{0, 1, 2, \ldots\}$. Prove that either $f(z) = 0$ for all $z \in \Omega$ or else it has at most m zeros in Ω.

COMPLEX ANALYSIS GRADUATE EXAM

FALL 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

1. Show that the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{4x^2 - \pi^2}\, dx$$

exists, and evaluate it.

2. Suppose that f is analytic on the open unit disc D and continuous on $\overline{D}$, and is non-constant on D. Suppose also there exist constants $0 < a \leq b$ so that $a \leq |f(z)| \leq b$ for $|z| = 1$.

(i) Show that $f(D) \subset B(0, b)$.

(ii) Show that either $f(D) \supset B(0, a)$ or $f(D) \cap B(0, a) = \emptyset$.

3. Suppose that entire functions f and g satisfy $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists $\lambda \in \mathbb{C}$ such that $f = \lambda g$.

4. Let Ω be the region between the circles $C_1 : |z| = 1$ and $C_2 : |z - 1/2| = 1/2$. Find a conformal mapping of Ω onto the open unit disc. [You may give your answer as the composition of several mappings, so long as each mapping is precisely described.]

COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2006

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx,$$

being careful to justify your methods.

2. Let G be the family of all analytic mappings from the half space $\{z : \operatorname{Re} z > 0\}$ into the open unit disc $\{z : |z| < 1\}$. Define $\theta = \sup\{|f'(\pi)| : f \in G\}$.

(i) Find an upper bound for θ.

(ii) Show that there exists $g \in G$ such that $|g'(\pi)| = \theta$.

3. Suppose f is analytic on $0 < |z| < \rho$.

(i) Give the definitions of the three possible types of isolated singularity of f at 0.

(ii) Prove that e^f cannot have a pole at 0.

4. Let $B(r)$ denote the open disc of radius r centered at 0, and let $f : B(1) \to B(1)$ be a holomorphic map.

(i) Assume that $f(0) \in (-1, 0)$. Show that $1/2 \notin f(B(1/2))$.

(ii) Assume instead that $f(B(1/2)) \supset B(1/2)$. Show that there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all $z \in B(1)$.

Complex analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

1. Let a be a real number with $a > e$. Show that the equation $e^z = az^n$ has n solutions inside the unit circle.

2. Find the Fourier transform $\hat{f}(\omega)$ of the function $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2}\, dx.$$

3. Fix $\tau \in \mathbb{C}$ with non-zero imaginary part. Let $f(z)$, $z \in \mathbb{C}$ be a non-constant meromorphic function such that $f(z) = f(z + m + n\tau)$ for all $m, n \in \mathbb{Z}$ (such functions are called doubly periodic). Show that f has infinitely many singularities.

4. Let $f(z)$ be a one to one conformal map from the unit disk to a square with center 0 such that $f(0) = 0$. Show that $f(iz) = if(z)$. Show also that if $f(z)$ is given by the power series $f(z) = \sum_{n=1}^{\infty} c_n z^n$ then $c_n = 0$ for all n such that $n - 1$ is not divisible by 4.

COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2007

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

Note that some problems are worth 10 points and others are worth 7 points.

1. (10 points) Let $\Omega \subset \mathbb{C}$ be a convex domain and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function satisfying $\mathrm{Re}(f'(z)) \geq 0$ for all $z \in \Omega$. Prove that f is injective on Ω.

2. (10 points) Let f be an analytic function on D satisfying $|f(z)| \leq 1$ for all $z \in D$ and having at least two fixed points z_1 and z_2. Show that $f(z) = z$ for all $z \in D$.

3. (10 points) Find a conformal mapping of the semicircular region $R = \{z : \mathrm{Im}(z) > 0, |z| < 1\}$ onto D. HINT: You may decompose this map into simpler ones.

4. (7 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 - 2x + 4}\, dx.$$

Justify your method.

5. (7 points) How many roots does the polynomial $p(z) = 2z^5 + 4z^2 + 1$ have in D? Justify your answer.

COMPLEX ANALYSIS GRADUATE EXAM

FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a > 1$. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that the Taylor series of f at 0 converges to f uniformly on $\mathbb{C}$ if and only if f is a polynomial.

3. Let f be a one-to-one holomorphic function on the unit disc $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $D = f(B_1)$ be the image of B_1 under f. Similarly let D_r be the image under f of the open disc $B_r = \{z \in \mathbb{C} : |z| < r\}$ for $0 < r < 1$. Show that if $h : D \to D$ is a holomorphic mapping leaving the point $f(0)$ fixed then

$$h(D_r) \subseteq D_r \text{ for } 0 < r < 1.$$

4. Let $f(z)$ be continuous in $\mathrm{Re}\, z \geq 0$ and analytic in $\mathrm{Re}\, z > 0$. Let $g(x)$ be continuous in $\mathrm{Re}\, z \leq 0$ and analytic in $\mathrm{Re}\, z < 0$. Assume that $f = g$ on $\mathrm{Re}\, z = 0$. Prove that f and g are differentiable in the closure of their domains and that $\partial f/\partial x = \partial g/\partial x$ at the origin.

COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a, b \geq 0$. Show that

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx = \pi(b-a).$$

Deduce that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi.$$

2. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{z}{n(1+nz^2)}$$

on the set $E = \{z = x + iy : 0 < y < x\}$.

3. (a) Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function on a connected open set D. Suppose $au(x,y) + bv(x,y) = c$ on D, where a, b and c are real constants which are not all zero. Show that f is constant on D.

(b) Let f and g be holomorphic functions in a connected open set D. Suppose that f and g have no zeros in D. Suppose also that there is a sequence a_n of points in D such that $a_n \to a \in D$ and $a_n \neq a$ for all n and such that

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)} \quad \text{for all } n.$$

Show that there is a constant c such that $f(z) = cg(z)$ in D.

4. Let $f(z)$ be a holomorphic function on the disc $|z| < 1$ and suppose that $f(0) = 0$. Show that the series

$$\sum_{n=1}^{\infty} f(z^n)$$

converges uniformly in any compact subset of this disc.

COMPLEX ANALYSIS GRADUATE EXAM
Fall 2008

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\{|z| < 1\} \setminus \{|z - \frac{1}{2}| < \frac{1}{2}\}$ conformally to the upper half plane.

(2) Evaluate the integral

$$\int_0^\infty \frac{1}{x^{1/2}(1 + x^2)}\, dx.$$

(3) Assume f is meromorphic in $\{|z| \leq 1\}$, and $|f(z)| = 1$ for all z with $|z| = 1$. Show that f is a rational function.

(4) A *fixed point* of a mapping f is a point z such that $f(z) = z$. Let $G = (0,1)^2$ be the open until square in $\mathbb{C}$. Show that if a holomorphic map $f : G \to G$ has two distinct fixed points, then it is the identity mapping.

(5) Let f be analytic in the unit disc, satisfying

$$M = \int_{\{|z|<1\}} |f(z)|\, dx\, dy < \infty.$$

Show that for all $|z| < 1$,

$$|f(z)| \leq \frac{M}{\pi(1 - |z|)^2}.$$

COMPLEX ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\Omega = \{|z| < 1\} \setminus [\frac{1}{2}, 1)$ conformally to the unit disk $D = \{|z| < 1\}$.

(2) Find a bounded harmonic function $\phi(x, y)$ in the region $\Omega = \{(x, y) : x^2 > y^2 + 1\}$ that satisfies the boundary condition $\phi(x, y) = 1$ if $x^2 = y^2+1,\ y > 0$ and $\phi(x, y) = -1$ if $x^2 = y^2+1,\ y < 0$.

(3) Evaluate the integral

$$\int_0^\infty \frac{\log^2 x}{1 + x^2}\, dx.$$

(4) Find the number of roots of the equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

in the annulus $\{1 < |z| < 2\}$.

COMPLEX ANALYSIS GRADUATE EXAM
Fall 2009

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\Omega = \{z = x + iy : |z| < 1, y > 1/\sqrt{2}\}$ conformally to the unit disk $D = \{z : |z| < 1\}$.

(2) Let $f(z)$ be an entire function so that $f(z)$ does not assume any value in $[0, \infty)$. Show that f is a constant.

(3) Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}$.

(4) Show that the infinite product $\displaystyle\prod_{k=0}^{\infty}(1+z^{2^k})$ converges for $|z| < 1$ and equals $\dfrac{1}{1-z}$ (for $|z| < 1$).

(5) Show that if $f : D \to D$ (D the unit disk) has two distinct fixed points, then $f(z) = z$.

COMPLEX ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Notation: $\Re z$ denotes the real part of the complex number z, and $\Im z$ its imaginary part.

(1) Map the region $\Omega = \{\Im z > 0\} \setminus \{iy : 0 < y \leq 1\}$ conformally to the unit disk $D = \{|z| < 1\}$.

(2) How many zeroes of $p(z) = z^4 + z^3 + 4z^2 + 2z + 7$ lie in the right half plane $\{\Re z > 0\}$?

(3) Let f be analytic in the unit disk $D = \{|z| < 1\}$ and continuous on its closure $\bar{D}$. Show that if f is real valued on the boundary $\partial D = \{|z| = 1\}$ then f must be a constant.

(4) By consideration of $\int e^{z+\frac{1}{z}}\, dz$, or otherwise, show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots .$$

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi(1-e^{-1})}{2}.$$

2. Suppose that f is holomorphic in a neighborhood of 0 and that

$$(*) \qquad \sum_{n=0}^{\infty} f^{(n)}(z)$$

is absolutely convergent at $z = 0$. Show that f is an entire function, and that $(*)$ is convergent for all $z \in \mathbb{C}$.

3. Let f be a non-negative real valued harmonic function in the disc $D = \{z \in \mathbb{C} : |z| < R\}$.

(i) Prove that

$$\frac{R-|z|}{R+|z|}f(0) \le f(z) \le \frac{R+|z|}{R-|z|}f(0) \quad \text{whenever } |z| < R.$$

[Hint: use the Poisson formula.]

(ii) Prove that

$$\frac{1}{3}f(0) \le f(z) \le 3f(0) \quad \text{whenever } |z| \le R/2.$$

(iii) Let K be a compact subset of the open disc D. Show that there is a constant M depending only on K and R such that

$$f(z_1) \le Mf(z_2) \quad \text{for all } z_1, z_2 \in K.$$

4. Liouville's theorem states that a bounded entire function f is constant.

i) Give a proof of Liouville's theorem. You may use standard results about holomorphic functions such as Cauchy's theorem and power series representation, but any result you use should be clearly stated

(ii) Suppose instead that f is entire and that $|f(z)| \le K(1+|z|^n)$ for some $K < \infty$ and positive integer n. Show that f is a polynomial of degree at most n.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2011

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1. Evaluate

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\, dx.$$

2. (i) Suppose that $u_1, u_2, \ldots, u_n$ and $u_1^2 + \cdots + u_n^2$ are harmonic functions on a connected open set D. Show that each function u_r $(1 \le r \le n)$ is constant.

(ii) A function $f : D \to \mathbb{C}$ with $f(x+iy) = u(x,y) + iv(x,y)$ is said to be complex harmonic if the real valued functions u and v are harmonic. Show that if $f(x+iy)$ and $(x+y)f(x+iy)$ are both complex harmonic then f is analytic.

3. Let $f : D \to D$ be an analytic function on a bounded domain D with $0 \in D$. Assume $f(0) = 0$ and $|f'(0)| < 1$. Let $F_n(z) = f \circ \cdots \circ f(z)$ (n times). Show that $F_n(z) \to 0$ as $n \to \infty$ uniformly on compact subsets of D.

[Hint: consider first the behavior of F_n on a small neighborhood of 0.]

4. Starting with the definition

"f is analytic on a set G if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in G$."

describe the sequence of intermediate results required to obtain the following theorem:

"Suppose f and g are both analytic on a connected open set G and there is a convergent sequence z_n with limit $z_\infty \in G$ such that $f(z_n) = g(z_n)$ for all n. Then $f = g$ on G."

[You do not need to prove any of the intermediate results, but you should give a brief indication of how each result is used to obtain the next one.]

COMPLEX ANALYSIS GRADUATE EXAM
Fall 2011

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(1) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta + 2\sin\theta}.$$

(2) Suppose the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Show that for $r < R$,

$$\int_{\{|z|=r\}} |f(z)|^2 \, dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(3) Let $f(z)$ be analytic on $\mathbb{C}$ and suppose that the line $\Gamma = \{t + it : t \in \mathbb{R}\}$ is mapped to itself, that is, $f(z) \in \Gamma$ for all $z \in \Gamma$. If $f(\sqrt{2}) = 3$, then what is $f(\sqrt{2}i)$?

(4) Let $\Omega \subset \mathbb{C}$, with $\Omega \neq \mathbb{C}$, be simply connected, and let $f : \Omega \to \Omega$ be a conformal bijection. If f has two distinct fixed points z_1, z_2 (that is, $f(z_1) = z_1, f(z_2) = z_2$), show that f is the identity map.