

## Topics for the Graduate Exam in Statistics 541a

Most of the following topics are normally covered in the course Math 541a

This is a one hour exam.

Parametric models, families of discrete and continuous distributions.

Exponential families, natural parameter space, Jensen, correlation, Holder, Markov and Chebyshev inequalities.

Order statistics, quartiles, percentiles, probability integral transformation and its inverse.

Multivariate normal Distribution, derived distributions from normal samples, t, chi squared, F. Mixtures.

Modes of convergence, limit theorems, Slutsky theorems. Delta method, variance stabilizing transformations.

Simple Random Sampling.

Sufficiency, completeness. Rao Blackwell and Lehman Scheffe Theorem.

Estimation, method of moments, maximum likelihood, unbiased estimation, Bayes estimation.

Comparison of estimators, optimality, Fisher information, Cramer Rao inequality, asymptotic efficiency.

Application of the EM algorithm, jackknife and bootstrap.

### References:

G. Casella and R.L. Berger, Statistical Inference

T.S. Ferguson, A Course in Large Sample Theory

P.J. Bickel and A. Doksum, Mathematical Statistics

E.L. Lehmann, Theory of Point Estimation

G.J. McLachlan and T. Krishnan, The EM Algorithm and Extensions

B. Efron, The Jackknife, the Bootstrap and Other Resampling Plans

W.R. Gilks, S. Richardson, and D.J. Spiegelhalter, Markov Chain Monte Carlo in Practice

# MATHEMATICAL STATISTICS (I) QUALIFYING EXAM (MATH 541A)

SPRING 2000

- (1) Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables having density

$$f(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{if } x \leq \theta \end{cases}$$

for some  $\theta \in (-\infty, \infty)$ .

- State the Factorization Theorem for sufficient statistics.
  - Find a one-dimensional sufficient statistic for  $\theta$ .
  - Find a 95% confidence interval for  $\theta$ .
  - Derive the likelihood ratio test of the null hypothesis that  $\theta \geq 0$  against that alternative that  $\theta < 0$ .
- (2) Outputs  $X_1, X_2, \dots, X_n$  from a physical device are independent and identically distributed random variables having exponential distribution with (unknown) mean  $\lambda^{-1}$ . A measuring device records the values of the  $X_j$  as long as  $X_j < c$  for some known threshold  $c > 0$ . If  $X_j \geq c$  then the device returns the value  $c$ . Define

$$S_n = \sum_{j=1}^n X_j I(X_j < c), \quad T_n = \sum_{j=1}^n I(X_j \geq c)$$

where  $I(A)$  denotes the indicator of the event  $A$ .

- Write down the likelihood function of the observed values in terms of  $S_n$  and  $T_n$ .
- Show that the Maximum Likelihood Estimator of  $\lambda$  is

$$\hat{\lambda} = \frac{n - T_n}{S_n + cT_n}$$

- Find the joint asymptotic distribution of  $(S_n, T_n)$ .

Hint:

$$\int_0^c x \lambda e^{-\lambda x} dx = \lambda^{-1}(1 - (1 + c\lambda)e^{-c\lambda})$$

and

$$\int_0^c x^2 \lambda e^{-\lambda x} dx = \lambda^{-1}(2 - (2 + 2c\lambda + c^2\lambda^2)e^{-c\lambda})$$

- Using the result of the previous part, or otherwise, find the asymptotic distribution of  $\hat{\lambda}$ .

# STATISTICS QUALIFYING EXAM (MATH 541A)

SPRING 2001

**Problem 1:** Let  $X \sim \mathcal{N}(\lambda\mu, v^2\Sigma)$ , where  $\mu \in \mathbb{R}^n$ , and  $\Sigma \in \mathbb{R}^{n \times n}$  are a known vector and positive definite matrix respectively, and  $\lambda \in \mathbb{R}$  and  $v^2 > 0$  are unknown parameters in  $\mathbb{R}$ .

- (a) Find the maximum likelihood estimators  $\hat{\lambda}$  and  $\hat{v}^2$  of  $\lambda$  and  $v^2$  on the basis of the observation  $X$ .
- (b) Determine whether or not  $\hat{\lambda}$  is unbiased for  $\lambda$ .
- (c) Calculate the variance of  $\hat{\lambda}$ .
- (d) *Demonstrate* what the estimators  $\hat{\lambda}$  and  $\hat{v}^2$  become when  $\mu = (1, 1, \dots, 1)$  and  $\Sigma$  is the identity matrix. Explain.

**Problem 2:** Let  $\theta > 0$  be unknown and suppose that  $(X, Y)$  is uniform over the triangular region with vertices at  $(0, 0)$ ,  $(\theta, 0)$ , and  $(0, \theta)$ . Let  $(X_i, Y_i)$  be iid as  $(X, Y)$ .

- (a) Find a one dimensional sufficient statistic  $T$  for  $\theta$ , and prove it is sufficient.
- (b) Find an unbiased estimate of  $\theta$  which is a function of  $T$ .
- (c) Is  $\hat{\theta}$  UMVU? Prove your claim.

# MATHEMATICAL STATISTICS QUALIFYING EXAM (MATH 541A)

FALL 2001

**Problem 1:** Let  $X_1, X_2, \dots, X_n$  be independently distributed random variables and  $X_i$  has an exponential distribution with mean  $i\mu$ , where  $\mu$  is unknown.

- (i) [10 points] Find a one dimensional sufficient statistic. Is that statistic complete? Why?
- (ii) [20 points] Find the maximum likelihood estimate  $\hat{\mu}$  for  $\mu$ . Show whether  $\hat{\mu}$  is an unbiased estimator of  $\mu$ .
- (iii) [20 points] Calculate  $\text{Var}(\hat{\mu})$ . Determine if it achieves the Cramer-Rao lower bound.

**Problem 2:** Let  $X_1, \dots, X_n$  be iid with mean  $\mu$  and variance  $\sigma^2$ .

- (i) [20 points] Using any method of your choice, show that

$$S^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j - \bar{X})^2$$

is an unbiased estimator of  $\sigma^2$ . Note that we **DO NOT** assume  $X_i$  to be normal.

- (ii) [10 points] Derive the following formula for the decomposition of the mean squared error of the estimator  $T = T(X_1, X_2, \dots, X_n)$  for the function  $q(\theta)$  of an unknown parameter  $\theta$

$$\text{MSE}_\theta(T) = \text{Bias}_\theta^2(T) + \text{Var}_\theta(T),$$

where  $\text{MSE}_\theta(T) = E_\theta(T - q(\theta))^2$  and  $\text{Bias}_\theta(T) = E_\theta T - q(\theta)$ .

- (iii) [20 points] Let  $X_1, \dots, X_n$  be iid normal  $\mathcal{N}(\mu, \sigma^2)$ ,  $\theta = (\mu, \sigma^2)$ . Compute the mean square error  $\text{MSE}_\theta(aS^2)$  for the estimation of  $\sigma^2$  and find the value of  $a$  which minimizes it. (Hint: If  $Z$  is standard normal  $N(0, 1)$ ,  $E(Z^4) = 3$ .)

February 2002

**Math 541ab Exam**

**Problem 1.**

Let  $X_1, \dots, X_n$  be iid with distribution

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

1. Find the maximum likelihood estimate  $\hat{\theta}_n$  of  $\theta$ .
2. Assuming  $n$  is large, find the approximate distribution of  $\hat{\theta}_n$ . Based on the approximate distribution, find an approximate 95% confidence interval for  $\theta$ .
3. Derive the likelihood ratio test for testing  $\theta = 1/2$  vs.  $\theta > 1/2$ .

**Problem 2.**

Let  $X_1, \dots, X_n$  be iid with distribution

$$f(x|\theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_1 + \theta_2} e^{-x/\theta_1}, & x > 0 \\ \frac{1}{\theta_1 + \theta_2} e^{x/\theta_2}, & x < 0, \end{cases}$$

where  $\theta_1, \theta_2 > 0$  are unknown.

1. Show that  $T_1 = \sum_{i=1}^n X_i I(X_i > 0)$  and  $T_2 = \sum_{i=1}^n X_i I(X_i < 0)$  are sufficient statistics.
2. Find the maximum likelihood estimates of  $\theta_1$  and  $\theta_2$ .
3. How will you test  $\theta_1 = \theta_2$  vs.  $\theta_1 \neq \theta_2$ ?

Math 541a Exam, continuation

3. Let  $\mathbf{X}$  be a multivariate normal random vector in  $\mathbf{R}^d$  with unknown mean vector  $\theta = (\mu_1, \dots, \mu_d)$  and *known* invertible covariance matrix  $\Sigma$ .

a) Show that the score function (derivative of the log density with respect to  $\theta$ ) is  $\Sigma^{-1}(\mathbf{X} - \theta)$ .

b) Calculate the Fisher information matrix  $\mathbf{I}(\theta)$ .

c) Calculate the Cramer Rao lower bound for the estimation of

$$g(\theta) = \frac{1}{2} \sum_{i=1}^d \mu_i^2$$

from an iid sample of  $n$  vectors with distribution as  $\mathbf{X}$ .

d) Compute the lower bound in c) when  $d = 2$  and  $\Sigma$  is the diagonal matrix with positive diagonal entries  $\sigma_1^2, \sigma_2^2$ .



Fall 2002 Math 541a Exam

1. From the set  $R = \{1, \dots, N\}$ , a set  $s_n$  of  $n$  numbers are chosen without replacement,  $0 < n < N$ .

a) Find the probability that a particular set  $s_n$  is chosen.

For  $i, j \in R, i \neq j$ , let  $\mathbf{I}_i$  and  $\mathbf{I}_j$  be the indicator that  $i$  and  $j$  are included in  $s_n$ , respectively.

b) Find the mean  $E\mathbf{I}_i$  and the covariance  $\text{Cov}(\mathbf{I}_i, \mathbf{I}_j)$ .

c) With given numbers  $a_1, \dots, a_N$ , let

$$W = \sum_{i \in s_n} a_i.$$

Calculate the mean and variance of  $W$ .

d) For what special case does  $W$  have the hypergeometric distribution? Find the mean and variance of the hypergeometric  $W$  using the formulas you derived in part b.

2. Let  $X_1, X_2, \dots, X_n$  be independently distributed with exponential density  $\frac{1}{\theta}e^{-x/\theta}$  for  $x > 0$ , and let the ordered  $X$ 's be denoted by  $Y_1 \leq Y_2 \leq \dots \leq Y_n$ , that is  $(Y_1, Y_2, \dots, Y_n) = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$ . Show that

1. The joint distribution  $(Y_1, Y_2, \dots, Y_r)$  is an exponential family with density

$$\frac{1}{\theta^r (n-r)!} \exp \left[ -\frac{\sum_{i=1}^r y_i + (n-r)y_r}{\theta} \right], \quad 0 \leq y_1 \leq \dots \leq y_r.$$

2. Let  $E_j$  be independent exponential random variables with mean  $1/j$ ,  $j = 1, 2, \dots$ , and set

$$Z_1 = E_n,$$

$$\begin{aligned}
Z_2 &= E_n + E_{n-1}, \\
&\vdots \\
&\vdots \\
&\vdots \\
Z_i &= \sum_{k=0}^{i-1} E_{n-k},
\end{aligned}$$

Show that  $(Z_1, Z_2, \dots, Z_r)$  has the same distribution as  $(Y_1/\theta, Y_2/\theta, \dots, Y_r/\theta)$ .

3. Based on the result in 2), show that

$$\frac{2(\sum_{i=1}^r Y_i + (n-r)Y_r)}{\theta}$$

is chi-square with  $2r$  degrees of freedom.



Math 541a Qualify Exam. Spring 2003

1. Let  $\hat{p}$  be the proportion of successes in  $n$  independent Bernoulli trials each having probability  $p$  of success.

a) Compute the expectation of  $\hat{p}(1 - \hat{p})$ .

b) Compute the approximate mean and variance of  $\hat{p}(1 - \hat{p})$  using the delta method.

2. Let  $(X_1, \dots, X_n) \in \mathbf{R}^n$  have density function  $p(\mathbf{x})$ .

a) Find the density function of the order statistics  $(X_{(1)}, \dots, X_{(n)})$ .

b) Use part a) to find the density of  $(U_{(1)}, \dots, U_{(n)})$ , the order statistics from a sample of i.i.d.  $\mathcal{U}[0, 1]$  variables.

c) Let  $E_1, \dots, E_{n+1}$  be i.i.d. exponential with density  $p(x) = e^{-x}\mathbf{1}(x > 0)$ , and for  $k = 1, \dots, n+1$  set

$$S_k = E_1 + \dots + E_k.$$

Show that

$$\left(\frac{S_1}{S_{n+1}}, \dots, \frac{S_n}{S_{n+1}}\right) =_d (U_{(1)}, \dots, U_{(n)}).$$

For parts d) and e), recall that for  $\alpha \in (0, 1)$  the  $\alpha$  quantile  $x_\alpha$  of the distribution  $F$  is a number which satisfies

$$F(x_\alpha) = \alpha.$$

d) For  $m$  an integer such that  $m/n \rightarrow \alpha$  as  $n \rightarrow \infty$ , take  $U_{(m)}$  as the estimate of  $x_\alpha$ ; for  $\mathcal{U}[0, 1]$  we have  $x_\alpha = \alpha$ . The error made in estimating  $x_\alpha$  is  $U_{(m)} - x_\alpha$ . Using part c), find the asymptotic distribution, as  $n \rightarrow \infty$ , of  $\sqrt{n}$  times this error.

e) Now let  $X_1, \dots, X_n$  be i.i.d. with density  $f(x) > 0$  for all  $x \in \mathbf{R}$ . Use d) to find the scaled asymptotic distribution of the error when using  $X_{(m)}$  to estimate the  $\alpha$  quantile of the distribution of the data.

**Fall 2003 Math 541a Exam**

1. For  $n \geq 1$ , let  $\mathbf{X}_n$  be a random vector in  $\mathbf{R}^n$ . For  $\Theta \subset \mathbf{R}$ , assume  $U_n(\mathbf{x}, \theta)$  is twice differentiable in  $\theta$ , and for  $\theta_0 \in \Theta$ , that there exist sequences  $a_n$  and  $b_n$  such that for some non-trivial random variable  $Y$  and some constant  $A \neq 0$

$$\begin{aligned} b_n U_n(\mathbf{X}, \theta_0) &\rightarrow_d Y, \quad a_n U'_n(\mathbf{X}, \theta_0) \rightarrow_p A, \quad \text{and} \\ a_n U''_n(\mathbf{X}, \theta) &= O_p(1) \quad \text{uniformly in some neighborhood of } \theta_0. \end{aligned}$$

(We say  $V_n(\theta) = O_p(1)$  uniformly in  $B$  if for every  $\epsilon \in (0, 1)$  there exists  $K$  such that for all  $n$ ,  $P(|V_n(\theta)| \leq K, \theta \in B) \geq 1 - \epsilon$ .)

a) If the solution  $\hat{\theta}_n$  to

$$U_n(\mathbf{X}_n, \theta) = 0,$$

is consistent for  $\theta_0$ , determine a sequence  $c_n$  and the distribution of a non-trivial  $Z$  such that

$$c_n(\hat{\theta}_n - \theta_0) \rightarrow_d Z.$$

b) What special case of a) corresponds to Maximum Likelihood Estimation?

2. Let  $T$  and  $C$  be independent Geometric random variables with success probability of  $r$  and  $s$ , respectively. That is

$$P[T = j] = r(1 - r)^{j-1}; j = 1, 2, \dots,$$

$$P[C = j] = s(1 - s)^{j-1}; j = 1, 2, \dots,$$

Let  $X = (\min(T, C), I(T \leq C))$ .

a) What is the distribution of  $X$ ? Calculate  $EX$ .

b) Let  $X(1), X(2), \dots, X(n)$  be a random sample from  $X$ . Denote  $X(i) = (X_1(i), X_2(i))$  and define

$$S_1 = \sum_{i=1}^n X_1(i) = \sum_{i=1}^n \min(T_i, C_i)$$

$$S_2 = \sum_{i=1}^n X_2(i) = \sum_{i=1}^n I(T_i \leq C_i).$$

What is the maximum likelihood estimate of  $(r, s)$ ,  $(\hat{r}, \hat{s})$  in terms of  $S_1$  and  $S_2$ ?

c) Using any methods of your choice, find the asymptotic covariance matrix of  $(\hat{r}, \hat{s})$ .



### Spring 2004 Math 541a Exam

1. Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of i.i.d. variables from the smooth density (or mass function)  $f(x, \theta)$ ,  $\theta \in \Theta \subset \mathbf{R}$ . Let  $L(\theta)$  be the likelihood, and  $I(\theta)$  the information of  $\mathbf{X}$  for  $\theta$ .

a) State the Cramér Rao Inequality.

b) Prove that an unbiased estimator  $\hat{\theta}$  of  $\theta$  achieves the Cramér Rao inequality if and only if

$$\frac{\partial \log L(\theta)}{\partial \theta} = I(\theta)(\hat{\theta} - \theta).$$

c) For

$$f(x; \sigma^2) = \frac{x}{\sigma^2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \quad x \geq 0,$$

determine the Cramér-Rao lower bound for  $\sigma^2$  and  $\sigma$  and demonstrate whether or not it is attainable in these two cases.

2. This problem is related to extreme value distribution.

a) Let  $Z$  be a standard normal random variable  $N(0, 1)$ . Prove that for  $t > 0$ ,

$$\sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} \exp\left(-\frac{t^2}{2}\right) \leq P\{|X| \geq t\} \leq \sqrt{\frac{2}{\pi}} \frac{1}{t} \exp\left(-\frac{t^2}{2}\right).$$

(Hint: For the left side inequality, note that, for  $x > t$ ,  $\frac{t^2}{1+t^2} < \frac{x^2}{1+x^2}$ . Note also that  $(\exp(-x^2/2)/x)' = -\frac{(1+x^2)\exp(-x^2/2)}{x^2}$ .)

b) Let  $X_1, X_2, \dots, X_n, \dots$  are independent identically distributed random variables with the same chi-square distribution with one degree of freedom. Find  $a_n$  and  $b_n$  such that  $a_n(\max_{1 \leq i \leq n} X_i - b_n)$  converges in distribution to a non-degenerate random variable.



Fall Math 541a Exam, Fall 2004

1. Let  $X_1, \dots, X_n$  be i.i.d. Geometric random variables with parameter  $p \in (0, 1)$ ,

$$P(X = x) = (1 - p)p^{x-1}, \quad x = 1, 2, \dots$$

- a. Find a complete sufficient statistic for  $p$ , and justify its properties.
- b. Compute the conditional distribution

$$P(X_1 = x_1, \dots, X_n = x_n | \sum_{j=1}^n X_j = x).$$

- c. Find the UMVUE (uniformly minimum variance unbiased estimator) for  $p$  when  $n = 1$ .

- d. Find the UMVUE for  $p$  in general.

2. Let  $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ , a multivariate normal vector in  $\mathbf{R}^n$ , with unknown mean vector  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$  and known covariance matrix  $\Sigma$ .

- a) Calculate the Fisher information of  $\mathbf{X}$  for  $\boldsymbol{\mu}$ . You may use that  $\partial \mathbf{y}' \mathbf{A} \mathbf{y} / \partial \mathbf{y} = 2 \mathbf{A} \mathbf{y}$ .

- b) Calculate the Fisher information in  $\mathbf{X}$  for the estimation of  $\sum_{i=1}^n \mu_i$ .

- c) Suppose that for some unknown scalar parameter  $\mu$  we have  $\boldsymbol{\mu} = \mathbf{a}\mu$  with  $\mathbf{a} \in \mathbf{R}^n$  a known vector. Calculate the Fisher information of  $\mathbf{X}$  for  $\mu$ .

- d) With  $\mathbf{a} = (1, \dots, 1)'$  in part b) and  $\Sigma = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$ , state and *justify* necessary and sufficient conditions on  $\sigma_1^2, \dots, \sigma_n^2$  such that there exists a consistent estimator of  $\mu$ .

e) For  $n = 2$ ,  $\mathbf{a} = (1, 1)'$  and  $\text{Var}(X_1) = \text{Var}(X_2) = 1$ , how should  $\Sigma$  be chosen to minimize  $\text{Var}(\hat{\mu})$  for the UMVU  $\hat{\mu}$  of  $\mu$ ?

**Math 541a Qualifying Exam, Spring 2005 (One-hour)**

1. Assume  $X_1, X_2, \dots$  are i.i.d. with uniform  $[0, 1]$  distribution. Let  $X_{(1)}, \dots, X_{(n)}$  be the order statistics from the smallest to the largest.
  - 1) Find the pdf (probability density function) of  $X_{(n)}$ ;
  - 2) Find the joint pdf of  $(X_{(1)}, X_{(n)})$ ;
  - 3) Find the pdf of  $X_{(1)} + X_{(n)}$ ;
2. Assume  $X_1, \dots, X_n$  are i.i.d. with pdf  $f(x|\theta) = \frac{\theta^2}{2}e^{-\theta^2|x|}, x \in \mathbb{R}$ , where  $\theta > 0$  is an unknown parameter.
  - (1) Find the Maximum Likelihood Estimator  $\hat{\theta}$  of  $\theta$  (not of  $\theta^2$ );
  - (2) Find the Fisher information  $I(\theta)$ ;
  - (3) Find the asymptotical distribution of  $\hat{\theta}$ .
3. Assume  $X_1, \dots, X_n$  are i.i.d. with pdf  $f(x|\alpha, \beta) = \frac{1}{\lambda(\alpha, \beta)}x^{\frac{\alpha}{\beta}-1}e^{-\frac{x^2}{2\beta}}, x > 0$ , where  $\lambda(\alpha, \beta) = \int_0^\infty x^{\frac{\alpha}{\beta}-1}e^{-\frac{x^2}{2\beta}}dx$  and  $\theta = (\alpha, \beta)$  lies in the parameter space
$$\Theta = \{(\alpha, \beta) : \alpha > 0, \beta > 0\}.$$
  - 1) Find the natural parameter space  $\Xi$ ;
  - 2) Find a complete and sufficient statistic (vector) for  $\theta$ ; and justify your answer;
  - 3) Find the UMVUE (uniformly minimum variance unbiased estimator) for  $\alpha$ ; and justify your answer.

**541a Qualifying Exam**

Fall, 2005

Name:

1	
2	
total	

1. Assume  $X_1, \dots, X_n$  are i.i.d. with p.d.f. (probability density function)

$$f(x|\theta) = \frac{2x}{\theta^2} \mathbf{1}(0 \leq x \leq \theta)$$

where  $\theta \in \Theta = (0, \infty)$  is an unknown parameter.

a) Find the Method of Moment Estimator  $\hat{\theta}_1$  of  $\theta$ , and its expectation and variance. Is it unbiased?

b) Find the Maximum Likelihood Estimator  $\hat{\theta}_2$  of  $\theta$  and its density function. Is it unbiased?

c) Find a complete sufficient statistic for, and the UMVU (uniformly minimum variance unbiased estimator) of,  $\theta$ .

d) Find constants  $a_n, b_n$  such that  $a_n(\hat{\theta}_1 - b_n)$  converges to a nondegenerate random variable in distribution, and give the limiting distribution.

e) Find constants  $c_n, d_n$  such that  $c_n(\hat{\theta}_2 - d_n)$  converges to a nondegenerate random variable in distribution, and give the limiting distribution.



2. Observations  $(X_1, X_2, \dots, X_n)$  are independently distributed over  $[0, 2\pi)$  according to the density  $f(x, \theta) = \exp[\theta_1 \cos x + \theta_2 \sin x - c(\theta)]$ , where  $\theta = (\theta_1, \theta_2) \in R^2$ .

a) Find an expression for the Fisher information  $I(\theta)$  (perhaps involving integrals) and show that it is positive definite for all  $\theta \in R^2$ .

b) Please reparameterize the model by  $\rho = \sqrt{\theta_1^2 + \theta_2^2}$ ,  $\cos \phi = \theta_1/\rho$ ,  $\sin \phi = \theta_2/\rho$ , and show that  $c(\theta)$  only depends on  $\rho$ .

c) Find in closed form an asymptotically efficient estimate for  $(\cos \phi, \sin \phi)$ .  
**Hint:**  $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ ,  $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ .

# 541a Qualifying Exam

Spring, 2006

Name:

1	
2	
total	

1. Let  $X_1, \dots, X_n$  be a random sample from a  $N(\theta, 1)$  distribution, where  $\theta \in \mathbf{R}$  is an unknown parameter.

a) Find the Fisher information  $I(\theta)$ .

b) Find a complete and sufficient statistic for  $\theta$ .

c) Find an unbiased estimate of  $P(X \leq 0)$ .

d) Find the UMVUE (uniformly minimum variance unbiased estimator) of  $P(X \leq 0)$ .

2. Let  $X_1, \dots, X_n$  be i.i.d. with  $P(X_i = 1) = 1 - P(X_i = 0) = p, p \in (0, 1)$ .

a) Show that the average,  $\bar{X}_n$ , is the MLE of  $p$

b) Find the mean  $\mu_n$  and variance  $\sigma_n^2$  of  $\bar{X}_n$ , and invoke the Central Limit Theorem to determine the asymptotic distribution of  $\bar{X}_n$ , properly centered and  $\sqrt{n}$  scaled.

c) Find the MLE  $\hat{\theta}$  of the log odds parameter,

$$\theta = \log \left( \frac{p}{1-p} \right)$$

and, under proper centering and  $\sqrt{n}$  scaling, apply the Delta Method to find its non-trivial asymptotic distribution.

d) Find a variance stabilizing transformation for the estimate of  $\hat{p}$  in a). (Recall that a variance stabilizing transformation for the estimator  $\hat{\theta}$  is a function  $g$  such that the (properly centered and scaled) non-trivial asymptotic distribution of  $g(\hat{\theta})$  does not depend on any unknown parameters.)

# Fall 2006 Math 541a Exam

- Let  $X_1, \dots, X_n$  be independent identically distributed samples taking values 0 or 1 with distribution

$$P(X_i = 1) = 1 - P(X_i = 0) = p.$$

Suppose that

$$T_n = \sum_{i=1}^n I(X_i = 1, X_{i+1} = 1)$$

is observed.

- Calculate the expectation  $T_n$ .
  - Find the method of moment estimator  $\hat{p}_n$  for  $p$  based on  $T_n$ .
  - Calculate the variance of  $T_n$ .
  - Find the (properly scaled and centered) non-trivial asymptotic distribution of  $\hat{p}_n$ .
- Suppose  $X_1, \dots, X_n$  are i.i.d. samples with density  $f(x, \theta)$ , where  $\theta \in E$ , an open set in  $\mathbf{R}$ . Suppose  $E_\theta(X_1) = \theta$ . Let  $\text{Var}_\theta(X_1) = \sigma^2(\theta) < \infty$ , be integrable in  $\theta$ . A transformation  $h: E \rightarrow R$  such that  $h' > 0$  is called variance stabilizing if for all  $\theta$ ,  $\sqrt{n}(h(\bar{X}) - h(\theta)) \rightarrow N(0, b^2)$ , where  $b$  is a constant.
    - Show that  $h$  is variance stabilizing if and only if  $h'(\theta) = b\sigma^{-1}(\theta)$  for some  $b$  and all  $\theta \in E$ .
    - Let  $X_1, \dots, X_n$  be the indicators of  $n$  binomial trials with probability of success rate  $\theta$ . Show that the only variance stabilizing transformation  $h$  such that  $h(0) = 0$ ,  $h(1) = 1$ , and  $h'(t) \geq 0$  for all  $t$ , is given by  $h(t) = (2/\pi) \sin^{-1}(\sqrt{t})$ .
    - (Continue) Let  $S = \sum_{i=1}^n X_i$  and  $\bar{X} = S/n$ . Show that  $\sin^{-1}(\sqrt{\bar{X}}) \pm z(1-0.5\alpha)/(2\sqrt{n})$  is an approximate level  $(1-\alpha)$  confidence interval for  $\sin^{-1}(\sqrt{\theta})$ , where  $z(\alpha)$  is the  $\alpha$ -th quantile of the standard normal distribution.



### Spring 2007 Math 541a Exam

1. Let  $\mathcal{P} = \{y_1, y_2, \dots, y_N\}$ , where  $y_i \in R$  are  $N$  distinct real numbers. The size  $N$  of  $\mathcal{P}$  may be very large and it is impractical to sample all the values of  $\mathcal{P}$ . Suppose that we are interested in the population average

$$\mu = \frac{1}{N} \sum_{i=1}^N y_i.$$

In a survey, a subset  $\mathbf{S} \subset \mathcal{P}$  of  $n$  elements,  $0 < n < N$ , are selected from  $\mathcal{P}$  without replacement and the values are recorded as  $X_1, X_2, \dots, X_n$ .

- (a) Describe the probability space and the resulting joint distribution of  $(X_1, X_2)$ .
- (b) Calculate the mean  $EX_1$  and the covariance  $\text{Cov}(X_1, X_2)$ .
- (c) Suppose that we use the sampling average

$$\hat{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

to estimate the population mean  $\mu$ . Show that  $\hat{X}$  is an unbiased estimator of  $\mu$ .

- (d) Show that

$$\text{Var}(\hat{X}) = \frac{N-n}{n(N-1)} V_y,$$

where

$$V_y = \frac{1}{N} \sum_{j=1}^N (y_j - \mu)^2.$$

2. Suppose that  $X = (X_1, X_2, \dots, X_n)$  follows a first order autoregressive model

$$X_t - \mu = \rho(X_{t-1} - \mu) + \epsilon_t, \quad t = 1, 2, 3, \dots, n,$$

where  $\mu \in R$  and  $\rho \in (-1, 1)$  are unknown and  $\epsilon_t$ 's are iid from  $N(0, 1)$ . Let  $\theta = (\mu, \rho)$ . Suppose  $X_0 = 0$ .

- (a) What is the joint density function for  $(X_1, X_2, \dots, X_n)$ .

- (b) Find the maximum likelihood estimator for  $(\mu, \rho)$ .
- (c) Calculate Fisher's information matrix and a lower bound for the variance of an unbiased estimator of  $\mu^2$ .

# Fall 2007 Math 541a Exam

1. (a) Show that for all  $\alpha > 0, \beta > 0$

$$G(x; \alpha, \beta) = e^{-\beta e^{-\alpha x}} \quad \text{for } x \in \mathbf{R}$$

is a distribution function (Gumbel family), and find its density and moment generating function.

- (b) Is  $G(x; \alpha, \beta)$  a member of the exponential family? Find sufficient statistics for  $X_1, \dots, X_n$ , an i.i.d. sample from  $G(x; \alpha, \beta)$ .
- (c) Let  $X_1, X_2, \dots$  be independent and identically distributed random variables with distribution function  $F(x)$ . Find the distribution function of

$$X_{(n)} = \max_{1 \leq i \leq n} X_i \quad \text{and} \quad X_{(1)} = \min_{1 \leq i \leq n} X_i.$$

- (d) Let  $G_1, \dots, G_n$  be i.i.d. with distribution  $G(x; \alpha, \beta)$  for some positive  $\alpha$  and  $\beta$ . Show that the largest variable  $G_{(n)} = \max_{1 \leq i \leq n} G_i$  is of the same family, and determine its parameters.
- (e) Taking the distribution

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

in part a, find a sequence  $b_n$  such that  $X_{(n)} - b_n$  converges to the Gumbel distribution as  $n \rightarrow \infty$ , and determine the parameters of the limit distribution.

2. Let  $(X_i, Y_i)$ ,  $i = 1, \dots, n$  be independent where  $X_i$  has an exponential distribution  $\mathcal{E}(\lambda_i)$  with density  $p(x, \lambda_i) = \lambda_i e^{-\lambda_i x}$ ,  $x > 0$ , and  $Y_i$  is independent of  $X_i$  with the exponential distribution  $\mathcal{E}(\theta \lambda_i)$ ,  $\theta > 0$ .

- (a) Show that the maximum likelihood estimates of  $(\lambda_1, \dots, \lambda_n, \theta)$  are

$$\hat{\lambda}_i = \frac{2}{X_i + \hat{\theta} Y_i}, \quad i = 1, \dots, n$$

and  $\hat{\theta}$ , which uniquely solves  $g(\theta) = 0$  where

$$g(\theta) = \frac{1}{n} \sum \left( \frac{X_i}{X_i + \theta Y_i} - \frac{1}{2} \right).$$

- (b) Show that the Fisher information bound for  $\theta$  under the assumption that  $\lambda_1, \dots, \lambda_n$  are known is  $\theta^2/n$
- (c) Show that  $U_i = X_i/(X_i + \theta Y_i)$ ,  $i = 1, \dots, n$  are independent uniform  $\mathcal{U}(0, 1)$ .
- (d) Now suppose  $\hat{\theta} \xrightarrow{p} \theta$ . Show that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, 3\theta^2)$$

by using the Taylor expansion

$$g(\hat{\theta}) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + o_p(\hat{\theta} - \theta).$$

### Spring 2008 Math 541a Exam

1. Let  $X_1, \dots, X_n$  be i.i.d. Poisson with mean  $\lambda$ .

Note that

$$p_0 = P(X_i = 0) = e^{-\lambda},$$

so if  $Y = \#\{i : X_i = 0\}$ , then  $\lambda$  might be estimated by

$$\tilde{\lambda} = -\log(Y/n).$$

- (a) What is the distribution of  $Y$ ?
  - (b) Use Taylor series to find approximations of  $E(\tilde{\lambda})$  and  $\text{Var}(\tilde{\lambda})$ .
  - (c) Compare (your approximation of)  $\text{Var}(\tilde{\lambda})$  with the variance of the MLE  $\hat{\lambda}$  of  $\lambda$ .
  - (d) Determine the Fisher information in the sample, and whether the MLE  $\hat{\lambda}$  achieves the Cramer-Rao lower bound.
2. Let  $X_1, \dots, X_n$  be i.i.d.  $N(\mu, \sigma^2)$  and  $\phi$  be the standard Gaussian density, and define the average absolute deviation, or AAD, by

$$\text{AAD} = \frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}|.$$

- (a) Show that

$$\sqrt{n}\{\text{AAD} - 2\sigma\phi(0)\} \longrightarrow N\left(0, \sigma^2\left(1 - \frac{2}{\pi}\right)\right).$$

- (b) Let  $\text{med}(\mathbf{X})$  be the median of  $\mathbf{X} = (X_1, \dots, X_n)$ , and  $\text{sign}(x) = -1, 0, 1$  if  $x$  is negative, zero or positive, respectively. Assuming

$$\text{med}(\mathbf{X}) = \mu + \frac{1}{2n\phi(0)} \sum_{i=1}^n \text{sign}(X_i - \mu) + o_p(n^{-1/2}),$$

and using the bivariate CLT, deduce that

$$(\sqrt{n}(\text{med}(\mathbf{X}) - \mu), \sqrt{n}(\text{AAD} - \sigma\sqrt{\frac{2}{\pi}})) \longrightarrow N_2\left(0, \sigma^2 \begin{bmatrix} \pi/2 & 0 \\ 0 & 1 - 2/\pi \end{bmatrix}\right).$$



**Hint:** For part a, show that

$$|\frac{1}{n} \sum |X_i - \bar{X}| - \frac{1}{n} \sum |X_i - \mu|| = o_p(n^{-1/2})$$

by assuming (without loss of generality) that  $\mu < \bar{X}$ , verifying

$$\begin{aligned} & |\sum \{|X_i - \bar{X}| - |X_i - \mu|\}| \\ = & |\bar{X} - \mu|(|\sum \{\mathbf{1}(X_i \leq \mu) - \mathbf{1}(X_i > \mu)\}| + 2 \sum \mathbf{1}(\mu < X_i \leq \bar{X})|), \end{aligned}$$

and then applying the law of large numbers.

The following fact may also be useful:  $E|X_1 - \mu| = 2\sigma\phi(0) = \sqrt{\frac{2}{\pi}}\sigma$ .

**Fall 2008 Math 541a Exam**

1. Let  $p \in (0, 1)$  and  $q = 1 - p$ .

(a) Show that

$$P(X = -1) = p \quad \text{and} \quad P(X = k) = q^2 p^k, \quad k = 0, 1, \dots$$

defines a probability distribution for the random variable  $X$ .

- (b) Given the single observation  $X$ , the statistic  $X$  is sufficient; is  $X$  also complete?
- (c) Determine all unbiased estimators of  $p$ , given one observation of  $X$  from the family above. Hint: Consider  $T(X) = \mathbf{1}(X = -1)$ .
- (d) Find the UMVU of  $p$ , or prove that it does not exist.
2. Consider the Pareto distribution  $P(a, c)$ , with positive parameters  $a$  and  $c$ , whose density function is given by

$$p(x; a, c) = \frac{ac^a}{x^{a+1}} \quad \text{for } x \geq c.$$

- (a) Verify  $p(x; a, c)$  is a density function, and find the associated distribution function.
- (b) When  $X$  has density  $p(x; a, c)$ , determine the distribution of  $Y = \log X$ .
- (c) Let  $X_1, \dots, X_n$  be a random sample from the Pareto  $P(a, c)$  distribution. Find the maximum likelihood estimators  $\hat{a}$  and  $\hat{c}$  of  $a$  and  $c$ , respectively.
- (d) Determine the distribution of  $\hat{c}$  or of  $2na/\hat{a}$ .

### Spring 2009 Math 541a Exam

1. Let  $X_1, \dots, X_n$  be a random sample from a Bernoulli distribution with parameter  $p \in (0, 1)$ , that is,

$$P(X_i = 1) = p, \quad \text{and} \quad P(X_i = 0) = 1 - p.$$

- (a) Find a complete sufficient statistic  $T_n(X_1, \dots, X_n)$  for  $p$ .
- (b) Justify  $T_n(X_1, \dots, X_n)$  is sufficient and complete using the definitions of sufficiency and completeness.
- (c) Find the maximum likelihood estimator (MLE) of  $p$ , and determine its asymptotic distribution by a direct application of the Central Limit Theorem.
- (d) Calculate the Fisher information for the sample, and verify that your result in part (c) agrees with the general theorem which provides the asymptotic distribution of MLE's.
- (e) Find a variance stabilizing transformation for  $p$ , that is, a function  $g$  such that

$$\sqrt{n}(g(\hat{p}) - g(p)) \rightarrow_d \mathcal{N}(0, \sigma^2)$$

where  $\sigma^2 > 0$  and does not depend on  $p$ ; identify both  $g$  and  $\sigma^2$ .

2. (a) Let  $\theta$  have a Gamma  $\Gamma(\alpha, \beta)$  distribution with  $\alpha, \beta$  positive,

$$p(\theta; \alpha, \beta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)},$$

and suppose that the conditional distribution of  $X$  given  $\theta$  is normal with mean zero and variance  $1/\theta$ .

Show that the conditional distribution of  $\theta$  given  $X$  also has a Gamma distribution, and determine its parameters.

- (b) Conditional on  $\theta$  as in part (a), suppose that a sample  $X_1, \dots, X_n$  is composed of independent variables, normally distributed with mean zero and variance  $1/\theta$ . Find the conditional expectation  $E[\theta|X_1, \dots, X_n]$  of  $\theta$  given the sample, and show that it is a consistent estimate of  $\theta$ , that is, that

$$E[\theta|X_1, \dots, X_n] \rightarrow_p \theta \quad \text{as } n \rightarrow \infty.$$

**Fall 2009 Math 541a Exam**

1. Let  $X_1, \dots, X_n$  be a random sample from a Bernoulli distribution with parameter  $p \in (0, 1)$ , that is,

$$P(X_i = 1) = p, \quad \text{and} \quad P(X_i = 0) = 1 - p.$$

- (a) Determine the UMVU of

$$q(p) = p(1 - p).$$

- (b) Prove that the odds ratio

$$q(p) = \frac{p}{1 - p}$$

is not unbiasedly estimable.

- (c) Determine necessary and sufficient conditions on  $q(p)$  such that the UMVU of  $q(p)$  exists.

2. Let  $Y_1, \dots, Y_n$  be independent with distribution  $Y_i \sim \mathcal{N}(\theta_0 + \theta_1 x_i, 1)$ , where  $x_1, \dots, x_n$  are known real numbers.

- (a) Determine the maximum likelihood estimator  $(\hat{\theta}_0, \hat{\theta}_1)$  of  $(\theta_0, \theta_1)$ .
- (b) Calculate the Fisher information matrix for  $(\theta_0, \theta_1)$ .
- (c) Compare the Cramer Rao lower bound for the estimation of  $\theta_1$  when  $\theta_0$  is unknown to the case where  $\theta_0$  is known, and show this second lower bound is the smaller.
- (d) With both parameters unknown, find a simple necessary condition on a sequence of real numbers  $x_1, x_2, \dots$  such that  $(\hat{\theta}_0, \hat{\theta}_1)$  is consistent for  $(\theta_0, \theta_1)$ .

### Spring 2010 Math 541a Exam

1. Let  $X_1, \dots, X_n$  be i.i.d.  $\Gamma(p, 1/\lambda)$  with density  $g_\theta(x) = \frac{1}{\Gamma(p)} \lambda^p x^{p-1} e^{-\lambda x}$ ,  $x > 0$ ,  $\theta = (p, \lambda)$ ,  $p > 0$ ,  $\lambda > 0$ .
  - (a) Find a moment estimate of the parameter.
  - (b) Show that the moment estimates,  $\tilde{\theta}$  are asymptotically bi-variate normal and give their asymptotic mean and variance covariance matrix.
  - (c) Compute the asymptotic variance covariance matrix of the maximum likelihood estimates. You may leave your answer in terms of  $\Gamma$  function derivatives.
2. Recall that the  $t$ -distribution with  $k > 0$  degrees of freedom, location parameter  $\ell$ , and scale parameter  $s > 0$  has density

$$\frac{\Gamma((k+1)/2)}{\Gamma(k/2)\sqrt{k\pi s^2}} \left\{ 1 + k^{-1} \left( \frac{x - \ell}{s} \right)^2 \right\}^{-(k+1)/2}.$$

Show that the  $t$ -distribution can be written as a mixture of Gaussian distributions by letting  $X \sim N(\mu, \sigma^2)$ ,  $\tau = 1/\sigma^2 \sim \Gamma(\alpha, \beta)$ , and integrating the joint density  $f(x, \tau|\mu)$  to get the marginal density  $f(x|\mu)$ . What are the parameters of the resulting  $t$ -distribution, as functions of  $\mu, \alpha, \beta$ ?

### Fall 2010 Math 541a Exam

1. Let  $\mu > 0$  be an unknown parameter, and suppose that  $X_1$  and  $X_2$  are independent random variables, each having the exponential distribution with density function

$$p(x; \mu) = \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right), \quad x > 0.$$

It is clear that since the distribution  $p(x; \mu)$  has mean  $\mu$  that an unbiased estimator of  $\mu$  is given by  $\bar{X} = (X_1 + X_2)/2$ .

- (a) Calculate the variance of the estimator  $\bar{X}$  of  $\mu$ .
- (b) Now consider the estimator of  $\mu$  given by  $T(X_1, X_2) = \sqrt{X_1 X_2}$ . Calculate the bias of  $T(X_1, X_2)$ .
- (c) Show that the mean square error of  $T(X_1, X_2)$  as an estimator of  $\mu$ , that is

$$\text{MSE}(T) = E(T(X_1, X_2) - \mu)^2,$$

is strictly smaller than the mean squared error of the estimator  $\bar{X}$  of  $\mu$ .

2. A random variable  $X$  has the Weibull( $\alpha, \beta$ ) distribution if

$$P_{\alpha, \beta}(X > x) = \exp\{-(x/\alpha)^\beta\} \quad \text{for } x \geq 0. \quad (1)$$

Suppose that  $X_1, \dots, X_n$  are i.i.d. Weibull( $\alpha_0, \beta_0$ ), where  $(\alpha_0, \beta_0)$  is in the interior of a compact parameter space  $\Theta \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ .

- (a) Show that the MLE of  $\alpha$  is given by

$$\hat{\alpha} = \left( n^{-1} \sum_{i=1}^n X_i^{\hat{\beta}} \right)^{1/\hat{\beta}}, \quad (2)$$

where  $\hat{\beta}$  is the MLE of  $\beta$ .

- (b) As an alternative to maximum likelihood, let  $\tilde{\beta}$  be any estimator of  $\beta$  and consider the estimator of  $\alpha$  given by

$$\tilde{\alpha} = \left( n^{-1} \sum_{i=1}^n X_i^{\tilde{\beta}} \right)^{1/\tilde{\beta}}. \quad (3)$$



Show that  $\tilde{\alpha}$  is a “pseudo-MLE” in the sense that it maximizes  $\ell(\alpha, \tilde{\beta})$ , where  $\ell(\alpha, \beta)$  is the log-likelihood function.

- (c) Prove that  $EX_1^{\beta_0} = \alpha_0^{\beta_0}$ .
- (d) Prove that if  $\tilde{\beta}$  is any consistent estimator, than  $\tilde{\alpha}$  given by (3) is consistent for  $\alpha$ . *Hint:* Letting  $Y_n(\beta) = n^{-1} \sum_i X_i^\beta$ , argue that it suffices to prove that  $Y_n(\tilde{\beta}) \rightarrow \alpha^{\beta_0}$  in probability. Then, using the convexity of  $Y_n(\beta)$ , show that

$$\left| Y_n(\tilde{\beta}) - Y_n(\beta_0) \right| \rightarrow 0 \quad \text{in probability,}$$

and complete the argument by applying part (2c) and the law of large numbers.

### Spring 2011 Math 541a Exam

1. Let  $X_1, \dots, X_n$  be i.i.d. with distribution  $\mathcal{N}(\mu, \sigma^2)$  and  $n \geq 2$ .
  - (a) Find UMVU estimates  $\hat{\mu}$  and  $\hat{\sigma}^2$  of  $\mu$  and  $\sigma^2$ , respectively, and prove that they are such.
  - (b) Derive the marginal distributions of  $\hat{\mu}$  and  $\hat{\sigma}^2$ , and prove that these estimators are independent.
2. For  $\theta \in \mathbb{R}$  let  $X_1, X_2, \dots, X_n$  be independent continuous random variables, each having density function

$$p(x; \theta) = \exp(-(x - \theta))I\{x > \theta\},$$

where  $I(x) = 1$  if  $x > 0$  and  $I(x) = 0$  otherwise. Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the corresponding order statistics.

- (a) Find the joint density function of  $(X_{(1)}, X_{(2)})$ , and the marginal densities of  $X_{(1)}$  and  $X_{(2)}$ .
- (b) Show that
$$T = X_{(1)} - (n - 1)(X_{(2)} - X_{(1)})/n$$
is an unbiased estimator of  $\theta$ .
- (c) Find the maximum likelihood estimate of  $\theta$ .