Topics for the Graduate Exam in Real Analysis

Most of the following topics are normally covered in the course Math 525a.

This is a one hour exam.

Measures: Sigma-rings, sigma fields. Set functions and measures. Construction of measures over Euclidean n-space. Variation of signed measures. Hahn decomposition theorem. Absolute continuity. Mutually singular measures. Product measures. Regular measures. Measurable functions.

Integration: Definition and basic properties of integrable functions over an abstract measure space. The Riemann integral and its relation to the Lebesgue integral. Lebesgue's dominated convergence theorem and related results. Radon-Nikodym theorem. Fubini's theorem.

Convergence: Almost everywhere convergence, uniform convergence, almost uniform convergence, convergence in measure and in mean. Egoroff's theorem. Lusin's theorem.

Differentiation: Almost everywhere differentiable functions. Termwise differentiation and relations to limits of sequences. Bounded variation. Fundamental theorem of calculus.

Metric spaces: Topological properties, compactness, completeness, continuity of functions, contractive mapping theorem. Baire category theorem.

References:

W. Rudin, Real and Complex Analysis
Munroe, Introduction to Measure and Integration
E. Hewitt and K. Stromberg, Real and Abstract Analysis
P. Halmos, Measure Theory
R.P. Boas, A Primer of Real Functions
F. Riesz and B. Sz-Nagy, Functional Analysis
N. Dunford and J. Schwartz, Linear Operators

Spring 92

REAL ANALYSIS QUALIFYING EXAM

SPRING 1992

Problem 1 Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence in $L^1(\,d\mu)$ which converges a.e. to $f \in L^1(\,d\mu)$. Prove: $f_n \to f$ in $L^1(\,d\mu)$ iff $\int |f_n|\,d\mu \to \int |f|\,d\mu$.
Hint: Apply Fatou's lemma to $|f| + |f_n| - |f - f_n|$.

Problem 2 Let $\{f_n\}$ be a sequence of Lebesgue-measurable real-valued functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$$

Prove: there exists a subsequence of $\{f_n\}$ such that $\{f_{n_i}(x)\}$ converges to 0 for a.e. x.

Problem 3 Prove that Lebesgue measure λ on $\mathbb{R}$ is translation-invariant: if A is a Lebesgue-measurable subset of $\mathbb{R}$, then for each $u \in \mathbb{R}$, $u + A$ is also Lebesgue-measurable and $\lambda(u + A) = \lambda(A)$.

Problem 4 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **lower semi-continuous** provided

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

whenever $\lim_n x_n = x$. Show that every lower semi-continuous function is Borel measurable.

Problem 5 Show that the function φ defined by

$$\varphi(p) = \int_0^\infty x^p e^{-x}\,dx \quad (p \ge 0)$$

is well-defined and differentiable on $(0, \infty)$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1993

Problem 1 Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, \ldots$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

Problem 2 Prove or find a counterexample: Let D be a countable dense subset of $(0, 1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0, 1)$.

Problem 3 Let f be a non-constant meromorphic function which is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

Problem 4 How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2$$

Hint: consider the image of the imaginary axis.

Problem 5 Show that a function $f : (a, b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

Problem 6 Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Problem 7 Suppose f is a non-negative function which is Lebesgue integrable on $[0, 1]$, and $\{r_n : n = 1, 2, \ldots\}$ is an enumeration of the rational numbers in $[0, 1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$$

converges for a.e. $x \in [0, 1]$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1994

Problem 1 Evaluate $\int_0^\infty \frac{\log x}{1+x^2}\, dx$

Problem 2 Show that $[0,1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

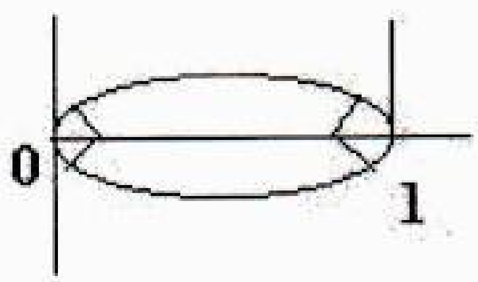
Problem 3 Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

$$|f(z)| \quad |f(0)|\frac{1+|z|}{1 \quad |z|}$$

Problem 4 Let $f : [1, +\infty) \to [0, +\infty)$ be Lebesgue measurable. Prove:

$$\int_1^\infty \frac{f(x)^2}{x^2} < +\infty \quad \Rightarrow \quad \int_1^\infty \frac{f(x)}{x^2}\, dx < +\infty$$

Problem 5 Map the region between the circular arcs (in the figure below) conformally to the unit disk. (Note that the top and bottom acrs intersect at right angles.)



Problem 6 Let $([0,1], \mathcal{A}, \ )$ denote the Lebesgue measure space on $[0,1]$. Give examples to show that for $f : [0,1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is imlied by, the condition "there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e."

Problem 7 An entire function is said to have **finite order** if there exists $c > 0$ such that $|f(z)| \quad \exp(|z|^c)$ for all $|z|$ sufficiently large; the **order** of f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $1/2$:

$$f(z) = \prod_{k=1}^{\infty} 1 \quad \frac{z}{k^2}$$

Problem 8 Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \ )$ with $\ (X) < \infty$. We say the sequence is **uniformly integrable** if

$$\lim_{R\to\infty} \sum_n \int_{|f_n| \quad R} |f_n|\, d \quad = 0$$

(a) Show that if there exists $g \in L^1(\ )$ such that $|f_n(x)| \quad g(x)$ for all x, n there the $\{f_n\}$ are uniformly integrable.

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\ )$ and

$$\lim_n \int f_n\, d \quad = \int f\, d$$

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1994

Problem 1 If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, must the derivative f' be continuous at SOME point of $\mathbb{R}$? Explain. (Write f' as the limit of a sequence of continuous functions.)

Problem 2 Evaluate

$$\int_{f(\Gamma_\rho)} \frac{\mathrm{d}z}{1+z}$$

where Γ_ρ is the square with verticies $\pm\rho$, $\pm i\rho$ and $f(z) = e^z$.

Problem 3 Let $(X, \mathcal{B}, v)$ be a finite measure space and f_n, f nonnegative bounded measurable functions on X. Define measures μ_n, μ by

$$\mu_n(A) = \int_A f_n \, \mathrm{d}v, \quad \mu(A) = \int_A f \, \mathrm{d}v$$

Prove: $f_n \to f$ in L^1 iff $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$ as $n \to \infty$.

Problem 4 Evaluate $\int_0^\infty \frac{x^{1/3}}{4+x^4} \, \mathrm{d}x$

Problem 5 Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0, 1]$ which converges in the L^2 sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the L^2 sense. Prove: $\{f_n\}$ converges to 0 uniformly.

Problem 6 Conformally map the open unit disk to a semi-infinite strip in the plane, $\{z \in \mathbb{C} : \Re z > 0, 0 < \Im z < a\}$ for some $a > 0$.

Problem 7 Suppose $(X, \mathcal{B}, \mu)$ is a finite measure space and f_n, g_n are measurable real-valued functions on X such that $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Show that given $\varepsilon > 0$ there exists an M such that $\mu(\{x \in X : |f_n(x)| > M\}) < \varepsilon$ for all n.

(b) Show that $f_n g_n \to fg$ in measure.

Spring 1995

Analysis Qualifying Examination
Tuesday, May 2, 1995

INSTRUCTIONS. Do **any seven** of the following problems; begin each problem on a fresh sheet of paper.

Problem 1. *Let $f(z)$ be analytic in $|z| \le 1$, with $f(0) = a_0 \ne 0$. If $M = \max_{|z|=1} |f(z)|$, show that $f(z) \ne 0$ for*

$$|z| < \frac{|a_0|}{|a_0| + M}.$$

Problem 2. *Let μ be a finite measure on $(X, \mathcal{B})$, suppose $f_n \to f$ a.e. on X, and $\|f_n\|_2 \le M < \infty$ for all n. Show that $f_n \to f$ in L^1.*

Problem 3. *Compute:*

$$\int_{|\xi|=1} \frac{d\xi}{\sqrt{\xi}}.$$

Problem 4. *Suppose μ is a measure on $(X, \mathcal{B})$, and suppose the function $f : \mathbb{R} \times X \to \mathbb{R}$ is differentiable in the L^2 sense, that is, $f(t, \cdot) \in L^2(\mu)$ for all t and $(f(t+h, \cdot) - f(t, \cdot))/h$ converges in $L^2(\mu)$ to a limit $g(t, \cdot)$ as $h \to 0$. Define*

$$\alpha(s,t) = \int_X f(s,x) f(t,x)\, d\mu(x).$$

Prove: $\dfrac{\partial^2 \alpha}{\partial t \partial s}$ exists.

Problem 5. *Show that $f(z) = z^5 + z^3 + 2z + 3$ has only one zero in the first quadrant $x \ge 0, y \ge 0$ (where $z = x + iy$).*

Problem 6. *Investigate the convergence of $\sum_n u_n$, where*

$$u_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

Problem 7. *If $f(z)$ is analytic for $|z| < 1$ and $f(0) = 0$, prove that*

$$\sum_{n=1}^{\infty} f(z^n)$$

converges in $|z| < 1$ to an analytic function.

Problem 8. *If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that its graph*

$$G = \{(x, f(x)) : x \in \mathbb{R}\}$$

has Lebesgue measure zero in $\mathbb{R}^2$. (Hint: first do it for bounded functions.)

Ken

Fall 1995

Graduate Exam in Analysis, Fall 1995

1. We say that "$f_n \to f$ almost in $L^1(\mu)$" if for all $\epsilon > 0$ there exists a set N such that $\mu(N) < \epsilon$ and $\int_{N^c} |f_n - f| d\mu \to 0$ as $n \to \infty$.

a) Show that if f_n converges to f almost in L^1, and f_n converges to g almost in L^1, then $f = g$ a.e.

Consider the following statements:

(1) f_n converges to f in L^1.

(2) f_n converges to f almost in L^1.

(3) f_n converges to f in measure.

b) Show that (1) implies (2) implies (3).

c) Show that neither of the two reverse implications in part b) hold.

2. Let μ be a measure on the Borel subsets of R^n. With τ denoting the collection of open subsets of R^n, we define the support of μ by

$$support(\mu) = \{x \in R^n : \mu(U) > 0 \text{ for all } U \in \tau \text{ with } x \in U\}.$$

Prove that the set $support(\mu)$ is closed.

3. If f_n is a sequence of continuous functions on $[0,1]$ with $f_n \to f$ a.e. m (Lebesgue measure), prove that for any $0 \le a < 1$, $[0,1]$ contains a compact subset K such that $m(K) > a$ and f is continuous on K. (Hint: Apply Egoroff's theorem.)

X 4. Let I be the collection of bounded open intervals (a,b) of R and m Lebesgue measure. Prove there is no Borel set E such that $m(A \cap E) = \frac{1}{2}m(A)$ for all $A \in I$.

5. Evaluate the following integral: $\int_0^\infty \frac{x \cos x}{x^2+1} dx$.

6. Conformally map the region $\{z = x + iy \in C : y > \frac{1}{4} - x^2\}$ to the unit disk $|z| < 1$.

7. Let $f(z)$ be a bounded analytic function on $|z| < 1$, and $\{z_n\}$ be the zeroes of $f(z)$, is it true that $\sum(1 - |z_n|) < \infty$?

Ma525a Qualifying Exam
Choose four of the following five questions.

1. Let f_n, f be real or complex valued measurable functions on the measure space $(X, \mathcal{M}, \mu)$. Suppose that f_n converges to f in measure; that is,

$$\forall \epsilon > 0 \quad \lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

a) Show that if there is a g such that $|f_n| \leq g$ a.e., then $|f| \leq g$ a.e.

b) Suppose that $\mu(X) < \infty$ and $|f_n| \leq g$ a.e with $\int g d\mu < \infty$. Show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

2. For a function $f : \mathbf{R} \to \mathbf{R}$, recall that

$$\limsup_{y\to x} f(y) = \lim_{\delta\to 0} \sup\{f(y) : |y - x| < \delta\}.$$

We say that f is *upper semi-continuous*, or u.s.c., if

$$\limsup_{y\to x} f(y) \leq f(x).$$

Prove that f is measurable if f is u.s.c.

3. Let ϕ be a measurable complex function on $\mathbf{R}$ satisfying

$$|\phi(x)| = 1 \quad \text{and} \quad \phi(x+y) = \phi(x)\phi(y) \quad \text{for all } x, y \in \mathbf{R}.$$

Prove that ϕ is continuous. (Hint: Show there exists a such that $A = \int_0^a \phi(t)dt \neq 0$ and consider $A^{-1}\int_0^a \phi(x+t)dt$.)

4. For $0 < \alpha < 1$, the sequence of numbers $\xi_n = \alpha^{n/(n+1)}2^{-n}$ satisfies $\xi_n > 2\xi_{n+1}$, $n = 0, 1, \ldots$. Let $B_0 = [0, 1]$. To obtain B_{n+1} given B_n, the union of intervals, remove from the middle of each subinterval of B_n the open interval of length $\xi_n - 2\xi_{n+1}$. Show that $B = \cap_{j=0}^{\infty} B_n$ is a closed, nowhere dense subset of $[0, 1]$ with measure α.

5. Let A be a bounded measurable subset of $\mathbf{R}$. Show that

$$\lim_{n\to\infty} \int_A \cos(nx)dx = 0.$$

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1996

Problem 1 (Stability of contractive iteration) Let (M, d) be a metric space, and suppose $T : M \to M$ satisfies

$$d(Tx, Ty) \leq k \cdot d(Tx, Ty) \quad \text{for all } \; x, y \in M$$

where $0 < k < 1$. Now suppose $\varepsilon > 0$, and a sequence $\{\hat{x}_n\}_{n=0}^{\infty}$ in M satisfies

$$d(\hat{x}_n, T\hat{x}_{n-1}) < \varepsilon \quad \text{for all } \; n \geq 1$$

Prove that for $0 \leq m < n$,

$$d(\hat{x}_m, \hat{x}_n) < k^n \frac{2d(\hat{x}_0, T\hat{x}_0)}{1-k} + \frac{2\varepsilon}{1-k}$$

Problem 2 How many zeros does the polynomial $p(z) = z^4 - 2z + 3$ have in the unit disk $|z| < 1$?

Problem 3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty} \varphi(x) f(x) \, \mathrm{d}x = 0$$

for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ which have compact support. Prove: $f(x) = 0$ for a.e. x.

Problem 4 Evaluate

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{2 + \sin\theta}$$

Problem 5 Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let M denote the space of Σ-measurable extended-real-valued functions on X. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, \mathrm{d}\mu$$

Show that ρ is a metric on M, and that $f_n \to f$ in the ρ-metric iff $f_n \to f$ in measure.

Problem 6 Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Prove that there exists a point $z_0 \in \mathbb{C}$ such that we can expand $f(z)$ into a power series about z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for which all $c_n \neq 0$.

Problem 7 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives f_{xy} and f_{yx}. Prove $f_{xy} \equiv f_{yx}$. **Hint:** use Fubini's theorem to integrate f_{xy} and f_{yx} over a rectangle $[a, b] \times [c, d]$.

Problem 8 Find a conformal mapping from the unit disk $|z| < 1$ to the region

$$\Omega = \{x + iy : (x < 0) \text{ and } (y > 0), \text{ or } (x \geq 0) \text{ and } (y > b)\}$$

where $b > 0$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1997

Directions: Do any **seven** of the following eight problems.

Problem 1 Prove: if $n \geq 2$ is an integer, then

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{x/n}{\sin(\pi/n)}$$

Problem 2 Suppose Ω is an open connected region of the complex plane and f is a non-constant analytic function on $\overline{\Omega}$. Prove: if $|f(z)| \equiv 1$ on the boundary of Ω, then $f(z)$ has at least one zero in Ω.

Problem 3 Formally, we have that

$$\begin{aligned}\frac{(-1)^n n!}{t^{n+1}} &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{t}\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^\infty e^{-tx}\,\mathrm{d}x \\ &= \int_0^\infty \frac{\partial^n}{\partial t^n} e^{-tx}\,\mathrm{d}x = \int_0^\infty (-1)^n x^n e^{-tx}\,\mathrm{d}x\end{aligned}$$

so that on setting $t = 1$ we obtain

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$$

Justify the calculation.

Problem 4 Let $X = C[0,1]$ be the space of all bounded continuous functions from $[0,1]$ to $\mathbb{R}$ with the sup-norm distance,

$$d(f,g) = \sup_{0\leq t\leq 1} |f(t) - g(t)|$$

You may assume that (X,d) is complete. Let $F : X \to X$ be a strict contraction, i.e., a function such that there exists $k < 1$ with

$$d(Fx, Fy) \leq k d(x,y) \quad \text{for all} \quad x, y \in X$$

Let I denote the identity operator on X, prove:

- $I + F$ is a 1-1 mapping of X onto X
- $(I+F)^{-1}$ is continuous

Problem 5 Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and let $\mathcal{F}$ be the family of all functions f on $[0,1]$ of the form

$$f(x) = \int_0^1 g(y)K(x,y)\,\mathrm{d}y$$

Problem 6 Show that for each $\varepsilon > 0$ the function

$$f(z) = \sin z + \frac{1}{z}$$

has infinitely many zeros in the strip $|\Im z| < \varepsilon$.

Problem 7 Determine the order of the entire function

$$f(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n^2}\right)$$

(Recall that the *order* of an entire function f is

$$\lim_{r\to\infty} \frac{\log\log M(r)}{r}$$

where $M(r) = \max_{|z|=r} |f(z)|$.)

Problem 8 Prove: if A and B are Lebesgue-measurable subsets of $\mathbb{R}$ with positive Lebesgue measure, then the set

$$A + B = \{a + b : a \in A, b \in B\}$$

has non-empty interior. (Hint: consider the convolution of the characteristic functions of A and B.)

Analysis Qualifying Exam, Fall 1997

Last Name:
First Name:
Social Security Number:

Please mark the 5 problems to be graded in the table on the right.

Problem	Points	Selection	Score
1	20	Y N	
2	20	Y N	
3	20	Y N	
4	20	Y N	
5	20	Y N	
6	20	Y N	
Total	100		

Problem 1. Let μ be a finite measure on $(X, \mathcal{B})$, and let α be a positive function on $\mathbb{R}$ such that $\alpha(x)/x \to \infty$ as $x \to \infty$. Suppose $f_n \to f$ a.e. on X, and $\|\alpha \circ f_n\|_1 \leq M < \infty$ for all $n \in \mathbb{N}$.

(a) Show that

$$\sup_n \|f_n \chi_{[f_n \geq K]}\|_1 \to 0, \quad \text{as} \quad K \to \infty.$$

(Here χ_A denotes the characteristic function of the set A.)

(b) Show that for each $K > 0$, $f_n \chi_{[f_n < K]} \to f \chi_{[f < K]}$ in L^1 as $n \to \infty$.

(c) Show that $f_n \to f$ in L^1 as $n \to \infty$.

Problem 2. Consider a nonnegative valued function $f \in L^1(\mu)$. Suppose $c \geq 0$, $A = \{x \in X : f(x) \geq c\}$, and $\mu(A) = \delta > 0$. Show that

$$\sup_{\{B:\mu(B)\leq\delta\}} \int_B f d\mu = \int_A f d\mu.$$

Problem 3. Show that for every $f, g \in L^1(\mathbb{R}, \mu)$, the following equality holds

$$\lim_{h \downarrow 0} \frac{1}{h} \left(\int |f + hg| d\mu - \int |f| d\mu \right) = \int_{\Sigma_0^c} g(x) \text{sign}(f(x)) d\mu + \int_{\Sigma_0} |g(x)| d\mu,$$

where μ is a measure on $\mathbb{R}$ and $\Sigma_0 = \{x : f(x) = 0\}$.

SPRING 1998 ANALYSIS QUALIFYING EXAM
MONDAY, MAY 4, 1998

DIRECTIONS. Do any seven of the following eight problems, **using the paper and pens provided**. Start each problem on a *fresh* sheet of paper. When you have completed the exam, be sure your name is printed on each page; sign the envelope, and return the exam papers in the envelope. You may keep this printed page.

Problem 1. Suppose $f \in L^1(d\mu)$. Prove: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set A with $\mu(A) < \delta$, there holds

$$\left| \int_A f \, d\mu \right| < \varepsilon.$$

Problem 2. Let f be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove: if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

Problem 3. Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ which converges in the L^1 sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the L^1 sense. Prove: $\{f_n\}$ converges to zero uniformly.

Problem 4. Suppose D is the open unit disk in $\mathbb{C}$, and $f : D \to D$ satisfies $f(1/2) = 1/2$. Show that $|f'(1/2)| \leq 3/4$.

Problem 5. Let $(X, \mathcal{T})$ be a topological space which has the property that every closed set F is the intersection of a *countable* family of open sets. Prove: any finite measure μ on the Borel field of $(X, \mathcal{T})$ is *regular*: for each Borel set E and each $\varepsilon > 0$, there exist an open set $G \supset E$ and a closed set $F \subset E$ such that $\mu(G \backslash F) < \varepsilon$. (Hint: consider the collection of Borel sets E for which this condition is true.)

Problem 6. Let D be the open unit disk in $\mathbb{C}$, and let $f : D \to D$ be analytic with $f(0) = 0$. Suppose

$$|f(z)| \geq \frac{1}{6} \quad \text{for all} \quad |z| = \frac{1}{4}.$$

Show that f assumes every value in the disk $|w| < \dfrac{1}{6}$.

Problem 7. Let $g : [0,1] \to \mathbb{R}$ be Lebesgue measurable, and suppose $f(x,y) := g(x) - g(y)$ is Lebesgue integrable on $[0,1] \times [0,1]$. Prove: g is Lebesgue integrable.

Problem 8. Evaluate: $\int_0^\infty \dfrac{\sqrt{x}}{1+x^3}\, dx$.

Q4: ... f analytic ... Show that $|f'(\frac{1}{2})| \leq 1$

announced 11:38am

PROBLEM 4. Suppose the measures μ_n, $n \geq 1$, on $(X, \mathcal{B})$ are uniformly absolutely continuous with respect to some finite measure ν, that is, given $\epsilon > 0$ there is $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu_n(A) < \epsilon$ for all n. Suppose also that $d\mu_n/d\nu \longrightarrow f$, ν-a.e. Show that there exists a measure μ such that $\mu_n(A) \longrightarrow \mu(A)$ for all measurable A. Identify the measure μ.

PROBLEM 5. Let μ be a finite measure on the Borel sets in a separable metric space X, and define $\text{supp}(\mu) = \{x \in X : \mu(D) > 0 \text{ for every open set D containing } x\}$ and $G = (\text{supp}(\mu))^c$.

(a) Show that G is open.

(b) Show that $\mu(G) = 0$.

(c) If B is open and $\mu(B) = 0$, show that $B \subset G$.

PROBLEM 6. Consider the following modes of convergence of functions $f_n \longrightarrow f$:

(i) almost everywhere
(ii) in measure
(iii) in L^1
(iv) almost uniformly
(v) uniformly.

(a) Which implications among these are valid in all measure spaces? You need not prove these; just list all of them or make a diagram.

(b) Let μ be counting measure on the positive integers. What additional implications, if any, are valid in this special case? (Prove these implications; you may use any of the implications in (a) without proof.)

Analysis Qualifying Exam

Spring, 1999

- In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.
- Start each problem on a fresh sheet of paper.

REAL ANALYSIS. **Do only three of the following four problems.**

1. Suppose f_n, where $n = 1, 2, \ldots$, and f are nonnegative functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Show that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for every measurable E. (Hint: Use Fatou's Lemma.)

2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $E \in \mathcal{M} \otimes \mathcal{N}$ (the product σ-algebra in $X \times Y$). Show that every section $E_x = \{y \in Y : (x, y) \in E\}$ is measurable.

3. Let A denote the set of all $f \in C[0,1]$ such that f is monotonic on some open subinterval of $[0,1]$. Show that A is meager (that is, of the first category) in $C[0,1]$ in the topology of uniform convergence.

4. (a) Show that the class of all step functions, of form $\sum_{j \leq n} c_j \chi_{(a_j, b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$ where μ is the Lebesgue measure on $\mathbb{R}$. (Hint: Why is the corresponding statement true for simple functions?)

   (b) Suppose $f \in L^1(\mu)$. Show that $\lim_{h \to 0} \int \left| f(x+h) - f(x) \right| dx = 0$. (Hint: Use (a).)

COMPLEX ANALYSIS. **Do all four problems.**

5. Suppose that f is analytic on $\mathbb{C}$ and that f is a homeomorphism of $\mathbb{C}$ onto a set U.

   (a) Show that f has a non-essential singularity at ∞.

   (b) Deduce that f must be of the form $f(z) = az + b$ for some $a \neq 0$ and that $U = \mathbb{C}$.

6. (a) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant M such that $\left| f^{(k)}(0) \right| \leq k^2 M^k$ for all $k \geq 1$. Show that f can be extended to be analytic on $\mathbb{C}$.

   (b) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant $M > 1$ such that $\left| f(1/k) \right| \leq M^{-k}$ for $k \geq 2$. Show that f is identically zero.

ANALYSIS QUALIFYING EXAM **FALL 1999**

In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other.

Start each problem on a fresh sheet of paper, and write on only one side of the paper.

REAL ANALYSIS. Answer question 1 and any two of the other three questions.

1. Let $\{f_n\}$ be a sequence of functions on $(X, \mathcal{A}, \mu)$. Suppose $\{f_n\}$ is Cauchy in measure, that is, for every $\varepsilon > 0$ there exists N such that $m, n \geq N$ implies

$$\mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) < \varepsilon.$$

Show that there exists f such that $f_n \to f$ in measure. HINT: For $n_1 < n_2 < \ldots$ you can write $f_{n_k} - f_{n_j} = \sum_{i=j+1}^{k}(f_{n_i} - f_{n_{i-1}})$.

2. Suppose $f : [0,1] \to \mathbf{R}$ is a nonnegative Lebesgue measurable function satisfying

$$\int f^n \, dm = \int f \, dm \quad \text{for all } n \geq 1.$$

Show that f is the characteristic function χ_E of some measurable $E \subset [0,1]$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let m denote Lebesgue measure on $[0,1]$. Suppose F_n and F map $X \times [0,1]$ into $\mathbf{R}$ and satisfy

(i) $F_n(x, \cdot)$ is absolutely continuous and nondecreasing for all n, x;

(ii) $F(x, \cdot)$ is continuously differentiable for all x;

(iii) $\frac{\partial}{\partial t} F_n(x,t) \to \frac{\partial}{\partial t} F(x,t)$ for almost every x, t;

(iv) $|F_n(x,t) - F_n(x,0)| \leq t g(x)$ for all n, x, t, where g is an integrable function on X.

Show that

$$\frac{\partial}{\partial t} \int_X F(x,t) \, d\mu(x) \bigg|_{t=0} = \int_X \frac{\partial F}{\partial t}(x,0) \, d\mu(x).$$

4. Let X be a metric space and μ a regular Borel measure on $(X, \mathcal{B})$ with $\mu(X) = 1$. Let $\mathcal{E} = \{F \in \mathcal{B} : F \text{ closed}, \mu(F) = 1\}$ and $H = \bigcap_{F \in \mathcal{E}} F$. (Note *regular* means $\mu(E) = \sup\{\mu(K) : K \text{ compact}, K \subset E\} = \inf\{\mu(U) : U \text{ open}, E \subset U\}$ for all $E \in \mathcal{B}$.)

(a) Show that $\mathcal{E}$ is closed under finite intersections.

(b) Show that $\mu(H) = 1$. HINT: Show that $\mu(H^c) = 0$.

COMPLEX ANALYSIS. See next page.

ANALYSIS QUALIFYING EXAM **SPRING 2000**

In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other.

Start each problem on a fresh sheet of paper, and write on only one side of the paper.

REAL ANALYSIS. Answer any three of the four questions.

1. Suppose that F_n, $n \geq 1$ are nondecreasing right-continuous functions on $\mathbf{R}$ and $F = \sum F_n$ is finite.

   (a) Show that F is right continuous.

   (b) Suppose $F_n' = 0$ a.e. for all n. Show that $F' = 0$ a.e.

[Hint: consider the corresponding measures μ_n, μ with $\mu_n((a,b]) = F_n(b) - F_n(a)$].

2. Let $(X, \mathcal{B}, \nu)$ be a finite measure space and f_n, f non-negative bounded measurable functions on X. Define measures μ_n, μ on $(X, \mathcal{B})$ by

$$\mu_n(A) = \int_A f_n \, d\nu \qquad \text{and} \qquad \mu(A) = \int_A f \, d\nu.$$

   (a) Show that if $f_n \to f$ in $L^1(\nu)$ then $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$.

   (b) Conversely show that if $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$ then $f_n \to f$ in $L^1(\nu)$.

3. Let μ^* be an outer measure on X and suppose Y is a μ^*-measurable subset of X. Let ν^* be the restriction of μ^* to subsets of Y. Show that a set $E \subset Y$ is ν^*-measurable if and only if E is μ^*-measurable.

4. Let $f, f_1, \ldots, f_n, \ldots$ be measurable functions from the measure space $(E, \mathcal{E}, \mu)$ to an open subset Ω of $\mathbf{R}^d$ such that for all $\varepsilon > 0$,

$$\mu\left(\{x \in E : \|f(x) - f_n(x)\| \geq \varepsilon\}\right) \to 0 \quad \text{as } n \to \infty.$$

Assume that μ is finite. Show that for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $\mu\left(\{x : f(x) \notin K\}\right) \leq \varepsilon$ and $\mu\left(\{x : f_n(x) \notin K\}\right) \leq \varepsilon$ for every n.

COMPLEX ANALYSIS. See next page.

REAL ANALYSIS QUALIFYING EXAM (MATH 525A)

FALL 2000

(1) Let μ be a finite Borel measure on $\mathbb{R}$ and let

$$f(x) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(y)}{|x-y|^{1/2}}$$

Here $\frac{1}{|x-y|^{1/2}}$ should be interpreted as $+\infty$ when $x = y$.

(a) Prove that f is finite a.e. with respect to Lebesgue measure on $\mathbb{R}$. HINT: Consider $[-M, M]$ in place of $\mathbb{R}$.

(b) Show that f need not be finite a.e. with respect to μ.

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions on X such that $\{f_n(x)\}$ is a Cauchy sequence for almost every x. Show that for each $\epsilon > 0$ there is a measurable $E \subset X$ and a finite M such that $\mu(X \setminus E) < \epsilon$ and $|f_n(x)| \leq M$ for all $x \in E$ and $n \geq 1$.

(3) Suppose $\{\mu_n\}$ is a sequence of finite measures on $(X, \mathcal{M})$ and $\mu_n \to \mu$ uniformly on $\mathcal{M}$, for some set function μ. Show that μ is countably additive. (Note: We don't *assume* μ is a measure.) HINT: For $E_1, E_2, \ldots$ disjoint and $k \geq 1$, consider $\mu(\cup_{i=\infty}^{\infty} E_i) - \sum_{i=1}^{k} \mu(E_i)$.

(4) Suppose μ_1, ν_1 are positive σ-finite measures on $(X_1, \mathcal{M}_1)$ and μ_2, ν_2 are positive σ-finite measures on $(X_2, \mathcal{M}_2)$, with $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Show that $\mu_1 \times \mu_2 \ll \nu_1 \times \nu_2$. (Here $\ll$ denotes absolute continuity.)

ANALYSIS QUALIFYING EXAM

MAY 10, 2001

Problem	Score
1	
2	
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Total	

Last Name: ____________________

First Name: ____________________

Social Security Number: ____________________

If you are taking the Real Analysis (525a) exam only: Do problems 1–4.

If you are taking the Complex Analysis (520) exam only: Do problems 5–8.

If you are taking both parts: In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.

Do as many problems as you can.

REAL ANALYSIS PROBLEMS

(1) Suppose f and g are absolutely continuous functions on an interval (a, b).
   (a) Show that f is bounded on (a, b).
   (b) Show that fg is absolutely continuous on (a, b).

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $\{f_n\}$ is a sequence of finite measurable functions with $f_n \to 0$ in measure.

(a) Show that

$$\int_X \frac{|f_n|}{1 + |f_n|}\, d\mu \to 0.$$

(b) Give an example to show (a) is false if we remove the assumption $\mu(X) < \infty$.

(3)(a) Suppose ν is a finite measure on $(X, \mathcal{M})$ with $\nu(X) > 0$, and ν is *atomless*, that is, every $E \in \mathcal{M}$ with $\nu(E) > 0$ has a subset $F \in \mathcal{M}$ with $0 < \nu(F) < \nu(E)$. Show that for every $\epsilon > 0$ there exists $A \in \mathcal{M}$ with $0 < \nu(A) < \epsilon$.

(b) Suppose μ is another finite measure on $(X, \mathcal{M})$, and $\alpha = \sup\left\{\frac{\mu(E)}{\nu(E)} : E \in \mathcal{M}, \nu(E) > 0\right\} < \infty$. Show that μ is absolutely continuous with respect to ν. HINT: Use (a).

(c) For $A, B \in \mathcal{M}$ we say A is a *ν-essential subset* of B if there exists $N \in \mathcal{M}$ with $\nu(N) = 0$, $A \backslash N \subset B$. Suppose there exists at least one set which achieves the supremum in (b). Show that there exists a measurable set Y such that $\mu(F) = \alpha\nu(F)$ if and only if F is a ν-essential subset of Y. HINT: Use (b).

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and let $g : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Show that the function

$$h(x) = \int_0^1 f(x - y) g(y) \, dy$$

is infinitely differentiable.

REAL ANALYSIS QUALIFYING EXAM (MATH 525A)

FALL 2001

(1) Suppose $(X, \mathcal{M})$ is a measurable space, μ is a positive measure with $\mu(X) < \infty$, and f is strictly positive measurable function. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf \left\{ \int_E f \, \mathrm{d}\mu : \mu(E) \geq \right\} > 0.$$

HINT: Consider a set where f is bounded away from 0.

(b) Show that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(2) Let μ be a finite positive measure on the Borel sets in a separable metric space X, and define $\mathrm{supp}(\mu) = \{x \in X : \mu(D) > 0 \text{ for every open set } D \ni x\}$ and $G = (\mathrm{supp}(\mu))^c$. Show that G is the largest open set with $\mu(G) = 0$.

(3) For a positive measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$, the *characteristic function* of μ is defined by

$$\phi(t) = \int \exp(itx) \, \mathrm{d}\mu(x), \qquad t \in \mathbb{R}.$$

(a) Suppose $\int |x| \, \mathrm{d}\mu(x) < \infty$. Prove that ϕ is continuously differentiable and $\phi'(0) = i \int x \, \mathrm{d}\mu(x)$.

(b) Compute the characteristic function of μ for $\mathrm{d}\mu(x) = \frac{1}{2}e^{-|x|} \, \mathrm{d}x$.

(4) Evaluate

$$\lim_{n\to\infty} \int_0^\infty \frac{e^{-x/n}}{1 + (x-n)^2} \, \mathrm{d}x$$

and justify your answer. HINT: The answer is not 0.

ANALYSIS QUALIFYING EXAM

FEBRUARY 13, 2002

Problem	Score
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2	
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Total	

Last Name:

First Name:

Social Security Number:

If you are taking the Real Analysis (525a) exam only: Do problems #1–4.

If you are taking the Complex Analysis (520) exam only: Do problems #5–8.

If you are taking both parts: In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.

Do as many problems as you can.

REAL ANALYSIS PROBLEMS

1. A mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ is said to be *Lipschitzian* if there exists a constant M such that

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

for all $x, y \in \mathbb{R}^N$. Prove: If $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitzian and $\Omega \subset \mathbb{R}^N$ is Lebesgue measurable, then $f(\Omega)$ is also Lebesgue measurable.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}$ a sequence of nonnegative $\mathcal{A}$-measurable functions which converges μ-a.e. to 0. Suppose there exists a finite constant M such that

$$\int \max\{f_1, \ldots, f_n\}\, d\mu \leq M$$

for all n. Prove: $\int f_n\, d\mu \to 0$ as $n \to \infty$.

3. Let $f \in L^1(X, \mathcal{A}, \mu)$. Prove:

$$\|f\|_1 = \int_0^\infty \mu\big(\{x : |f(x)| \geq \lambda\}\big)\, d\lambda.$$

4. Let f be continuous on $[-1,1]$. Show that

$$\lim_{n\to\infty} n \int_{-1/n}^{1/n} f(x)\big(1 - n|x|\big)\, dx$$

exists, and evaluate it.

Real Analysis Qualifying Exam
USC Department of Mathematics
September 26, 2002

Instructions. Do all of the following problems, *on separate pieces of paper.*

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every interval $[a, b]$, and that both f and f' are in $L^1(\mathbb{R})$. Prove:

$$\int_{-\infty}^{\infty} f'(x)\, dx = 0.$$

Problem 2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $\{f_n\}$, $\{g_n\}$ are sequences of measurable real-valued functions which converge in measure to f and g respectively.

(a) Prove: if $\mu(X) < +\infty$, then $\{f_n g_n\}$ converges in measure to fg.
(b) Show (by counterexample) that the hypothesis $\mu(X) < +\infty$ cannot be removed in part (a).

Problem 3. Prove: if $f \in L^1(0, 1)$ and $a > -1$, then the integral

$$f_a(x) = \int_0^x (x - t)^a f(t)\, dt$$

exists for almost every $x \in (0, 1)$, and that $f_a \in L^1(0, 1)$.

Problem 4. Let μ be a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ which is translation-invariant,

$$\mu(E + a) = \mu(E) \quad \text{for all Lebesgue-measurable sets } E \text{ and all } a \in \mathbb{R},$$

and that $\mu([0, 1]) < +\infty$. Prove that μ is a multiple of Lebesgue measure.

525 a

REAL ANALYSIS QUALIFYING EXAM
USC DEPARTMENT OF MATHEMATICS
FEBRUARY 12, 2003

INSTRUCTIONS. Do *four* out of the *five* problems, *on separate pieces of paper*. Be sure to justify your work.

Problem 1. $\{f_n\}$ is a sequence of measurable real-valued functions on a measure space $(X, \mathcal{A}, \mu)$.

(i) Suppose that $f_n \to f$ in measure and $|f_n| \leq g \in L^1(d\mu)$. Show that $f_n \to f$ in $L^1(d\mu)$.
(ii) Show that the result in (i) is false if the condition $|f_n| \leq g \in L^1(d\mu)$ is omitted.

Problem 2. For $a > 0$, show that

$$\int_0^\infty e^{-ax} x^{-1} \sin x \, dx = \arctan(a^{-1})$$

by integrating $e^{-axy} \sin x$ with respect to x and y.

Problem 3. Let A and B be Borel measurable subsets of a circle C of circumference 1 centered at the origin. Let A_t denote the set A rotated about the origin through an arc of length t. Prove that there exists a value of t such that

$$m(A_t \cap B) \geq m(A)m(B),$$

where m denotes the arclength measure.

Problem 4. Consider the functions $f_n(x) = n^\alpha x e^{-nx^2}$ to be integrated with respect to Lebesgue measure over the interval $E = [0, 1]$.

(i) Determine the values of the constant α for which the dominated convergence theorem applies.
(ii) Determine the values of α for which

$$\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n$$

Problem 5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitzian* if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Prove: if f is Lipschitzian and $\Omega \subset \mathbb{R}$ has Lebesgue measure zero, then the image $f(\Omega)$ has Lebesgue measure zero.

FALL 2003 REAL ANALYSIS (MATH 525A) QUALIFYING EXAM
WEDNESDAY, SEPTEMBER 24, 2003

DIRECTIONS. Do **exactly four** of the following five problems. Start each problem on a *fresh* sheet of paper, and write on only one side. When you have completed the exam, be sure your name is printed on each page. You may keep this printed page.

Problem 1. Let E be a Lebesgue-measurable subset of $\mathbb{R}$ which has the property that $x \in E, y \in E, x \neq y$ implies that $(x+y)/2$ is *not* in E. Prove: E has Lebesgue measure zero.
Hint: Show that for an interval (a,b), for a fixed $x_0 \in (a,b) \cap E$,

$$\frac{1}{2}x_0 + \frac{1}{2}\left((a,b) \cap E\right)$$

is a subset of (a,b) which has half the measure of $(a,b) \cap E$ and is disjoint from $(a,b) \cap E$. Conclude that the measure of $(a,b) \cap E$ therefore does not exceed $\frac{2}{3}(b-a)$.

Problem 2.

(a) Show that the class of all step functions, i.e. those of the form $\sum_{j=1}^{n} c_j \chi_{(a_j,b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$, where μ is Lebesgue on $\mathbb{R}$. [**Hint**: why is the corresponding statement true for simple functions?]

(b) Suppose $f \in L^1(\mu)$. Use the result in (a) to show that

$$\lim_{h \to 0} \int |f(x+h) - f(x)|\, dx = 0.$$

Problem 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be "lower semi-continuous" if

$$\liminf_{n \to \infty} f(x_n) \geq f(x) \quad \text{whenever } x_n \to x.$$

Prove that a lower semi-continuous function is Borel measurable.

Problem 4. Show that for $a > -1$

$$\int_0^1 x^a (1-x)^{-1} \ln x \, dx = -\sum_{k=1}^{\infty} (a+k)^{-2},$$

being careful to justify your calculations.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2}\, dt.$$

Prove: f is continuous on $\mathbb{R}$.

REAL ANALYSIS QUALIFYING EXAM
SPRING 2004

Answer all four questions. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos(x)}{x^{3/2}}\, dx$$

exists.

2. Suppose f_n, $n = 1, 2, \ldots$, and f are non-negative measurable functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and

$$\int_X f_n(x)\, d\mu(x) \to \int_X f(x)\, d\mu(x).$$

Show that

$$\int_X f_n(x)g(x)\, d\mu(x) \to \int_X f(x)g(x)\, d\mu(x)$$

for every bounded measurable function g. [Hint: use Fatou's Lemma].

3. Let $f \in L^1(\mathbf{R^d})$. Evaluate

$$\lim_{y\to\infty} \int_{\mathbf{R^d}} |f(x+y) - f(x)|\, dx.$$

[Note that the limit is NOT as $y \to 0$.]

4. Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that

$$|f(x) - f(x_0)| \le n|x - x_0| \quad \text{for all } x \in [0,1].$$

(i) Show that E_n is nowhere dense in $C([0,1])$.
(ii) Show that the set of nowhere differentiable functions $f \in C([0,1])$ is non-empty.

REAL ANALYSIS GRADUATE EXAM

FALL 2004

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n \, dx \right)^{1/n} = \sup_{x\in[a,b]} |f(x)|.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Prove that

$$g(x) = \sum_{n=1}^{\infty} f\left(2^n x + \frac{1}{n}\right)$$

is integrable and

$$\int_{-\infty}^{\infty} g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

3. A Lebesgue integrable function f defined on the interval $[0, 4]$ has the property that $\int_E f(x)\, dx = 0$ for all measurable E with $m(E) = \pi$. Must $f = 0$ a.e.?

4. Let $f(x) = x^2 \sin(1/x^2)$ and $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = g(0) = 0$.
   (i) Show that f and g are each differentiable everywhere (including $x = 0$).
   (ii) Show that $f \notin BV([-1, 1])$.
   (iii) Show that $g \in BV([-1, 1])$.

REAL ANALYSIS GRADUATE EXAM

SPRING 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. A function $f : [a, b] \to \mathbb{R}$ is said to be Hölder continuous of order α if there is a constant L such that

$$|f(x) - f(y)| \le L|x - y|^\alpha \qquad \text{for all } x, y \in [a, b].$$

(i) Show that if f is Hölder continuous of order 1 then it has bounded variation.

(ii) Let $\alpha \in (0, 1)$. Give an example of a function which is Hölder continuous of order α and does not have bounded variation.

(iii) Give an example of a function of bounded variation which is not Hölder continuous for any $\alpha > 0$.

2. Let $f \in L^1([0, 1])$. For $k = 1, 2, \ldots$ let f_k be the step function defined on $[0, 1]$ by

$$f_k(x) = k \int_{j/k}^{(j+1)/k} f(t)\, dt \quad \text{for } \frac{j}{k} \le x < \frac{j+1}{k}.$$

Show that f_k tends to f in L^1 as $k \to \infty$.
(Hint: Treat first the case when f is continuous, and use approximation.)

3. Suppose that f_n, $n = 1, 2, \ldots$, and f are complex valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Define the terms

(i) f_n converges in measure to f,

(ii) f_n is Cauchy in measure.

Show that if f_n is Cauchy in measure there exists a subsequence n_k and a measurable function g such that f_{n_k} converges to g almost everywhere, and f_n converges to g in measure.

4. Let $a_1, a_2, \ldots > 0$. Prove that $\sum_{i=1}^{\infty} a_i = \infty$ is a necessary and sufficient condition that there exists an enumeration of the rationals

$$\mathbb{Q} = \{r_1, r_2, \ldots\}$$

such that

$$\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}.$$

REAL ANALYSIS GRADUATE EXAM
Fall, 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $(X, \mathcal{B}, \mu)$ be a measure space with μ finite. For all $A \subset X$ define

$$\mu_1(A) = \sup\{\mu(B) : B \in \mathcal{B}, B \subset A\}, \qquad \mu_2(A) = \inf\{\mu(B) : B \in \mathcal{B}, B \supset A\}.$$

(a) Show that $\mu_1(A^c) + \mu_2(A) = \mu(X)$ for all $A \subset X$.

(b) Let $\mathcal{A} = \{A \subset X : \mu_1(A) = \mu_2(A)\}$. Show *directly from (a) and/or the definitions* that $\mathcal{A}$ is a σ-algebra.

(2) Suppose μ is a measure on $[0, \infty)$ satisfying

$$\int_{[0,\infty)} e^{ax} \, \mu(dx) < \infty \quad \text{for some } a \in \mathbb{R}.$$

Show that the function

$$\psi(t) = \int_{[0,\infty)} e^{tx} \, \mu(dx)$$

is infinitely differentiable on $(-\infty, a)$.

(3) Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $0 \leq f < \infty$ is a measurable function, and $d\nu = f d\mu$.

(a) Suppose μ is σ-finite. Show that ν is σ-finite. HINT: You must deal with the fact that f is not assumed bounded or integrable.

(b) A measure ρ is called *semifinite* if for every measurable set E with $\rho(E) > 0$, there is a measurable $F \subset E$ with $0 < \rho(F) < \infty$. Show that if $0 < f < \infty$ and ν is semifinite, the μ is semifinite. (We do not keep the assumption made in (a) that μ is σ-finite.)

(4) Suppose $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X)$ finite, $\{f_n\}$ are integrable functions, and $f_n \to f$ in L^1.

(a) Show that $f_n \to f$ in measure.

(b) Show that the measures $d\nu_n = |f_n| \, d\mu$ are uniformly absolutely continuous with respect to μ. NOTE: This means that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu_n(E) < \epsilon$ for all n. Absolute continuity of each ν_n *individually* is a standard result which you may make use of.

REAL ANALYSIS GRADUATE EXAM
Spring 2006

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of (i) and (ii) give a proof or a counterexample.

(i) Is it true that f is bounded on $\mathbb{R}$?

(ii) Is it true that $f(x) \to 0$ as $x \to \infty$?

How do the results for (i) and (ii) change under the additional assumption that f' exists everywhere and is bounded?

(2) For $y > 0$ define

$$G(y) = \int_0^\infty \frac{1 - e^{-yx^2}}{x^2}\, dx.$$

(a) Show that this integral is finite for all $y > 0$.

(b) Show that G is differentiable, and find an explicit formula for $G'(y)$ and $G(y)$. HINT: You may take as given that $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$.

(3) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f_n, f be real-valued measurable functions and suppose $f_n \to f$ a.e. Then there exists a partition of X into disjoint measurable sets $E_0, E_1, E_2, \ldots$ with $\mu(E_0) = 0$ and with $f_n \to f$ uniformly on E_i for each $i \geq 1$. HINT: Egoroff's Theorem requires a finite measure space.

(4) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf g(x_n) \geq g(x) \quad \text{whenever } x_n \to x.$$

(a) Suppose that $f_k, k = 1, 2, 3, \ldots$ is a sequence of continuous functions, and $f(x) = \sup_{k \geq 1} f_k(x)$ is finite for all x. Show that f is lower semi-continuous.

(b) Show that a lower semi-continuous function is measurable.

Real analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

1. Find necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to belong to the σ-algebra generated by all one-point subsets of $\mathbb{R}$.

2. Let (X, μ) be a measure space. Which of the following implications are true?

a. $\mu(X) < \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

b. $\mu(X) = \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

c. $\mu(X) < \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

d. $\mu(X) = \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

Give proof or counter-example in each case.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -e^{-(y-x)} & \text{if } x < y. \end{cases}$$

a. Is f Lebesgue integrable?

b. Is it true that

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\, dx \right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx \ ?$$

4. Let $f \in L^1(\mathbb{R})$. Show that for each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = f(x)(\sin x)^n$$

also belongs to $L^1(\mathbb{R})$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = 0.$$

REAL ANALYSIS GRADUATE EXAM

SPRING 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\{\mu_n\}$ be a sequence of measures on $(X, \mathcal{M})$ with $\mu_1(E) \leq \mu_2(E) \leq \ldots$ for all $E \in \mathcal{M}$. Let $\mu(E) = \lim_n \mu_n(E)$. Show that μ is a measure.

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $f \in L^1(\mu)$ is strictly positive. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf \left\{ \int_E f \, d\mu : \mu(E) \geq \alpha \right\} > 0.$$

(b) Show that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(3) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The *graph* of f is $\{(x, f(x)) : x \in [0,1]\}$. Show that the graph has two-dimensional Lebesgue measure 0.

(4) Let $n \geq 1$. Show that the function

$$g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x + 1} \, dx, \qquad u \in (0,1),$$

is differentiable in $(0,1)$.

REAL ANALYSIS GRADUATE EXAM

FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. In questions 3 and 4, $|A|$ is used to denote the Lebesgue measure of a measurable subset $A \subseteq \mathbb{R}^d$.

1. By differentiating the equation

$$\int_{-\infty}^{\infty} e^{-tx^2}\, dx = \sqrt{\frac{\pi}{t}} \qquad t > 0$$

show that

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

for $n \geq 1$. You should be careful to justify your calculations.

2. (a) Construct a sequence f_n of Lebesgue measurable functions on $(0, 1)$ such that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0, 1)$ and

$$\int_0^1 |f_n(x)|\, dx \to \infty$$

as $n \to \infty$.

(b) Give an example of a continuous function $F : [a, b] \to \mathbb{R}$ which is differentiable almost everywhere in $[a, b]$ with F' Lebesgue integrable on $[a, b]$ and such that

$$F(b) - F(a) \neq \int_a^b F'(t)\, dt.$$

3. Let g_k, $k = 1, 2, \ldots$, be a sequence of nonnegative measurable functions on a measurable subset E of $\mathbb{R}^d$. Suppose that

$$|\{x \in E : g_k(x) > 1/2^k\}| < 1/2^k \quad \text{for each } k \geq 1.$$

Prove that $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on E .

4. Let $g \in L^p(\mathbb{R}^d)$ and define

$$\mu(t) = |\{x \in \mathbb{R}^d : |g(x)| > t\}| \quad \text{for } t \geq 0.$$

Show that

$$\int_{\mathbf{R}^d} |g(x)|^p\, dx = -\int_0^{\infty} t^p\, d\mu(t).$$

REAL ANALYSIS GRADUATE EXAM

SPRING 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m denote Lebesgue measure on the unit square $E = [0,1] \times [0,1]$. In each case determine whether the integral exists:

$$\text{(i)} \quad \int_E \frac{1}{x-y}\,dm(x,y), \qquad \text{(ii)} \quad \int_E \frac{1}{x+y}\,dm(x,y).$$

2. Let f be a nonnegative measurable function on [0,1] such that $\int_0^1 f(x)\,dx = 1$. Define a measure μ on $[0,1]$ by

$$\mu(A) = \int_A f(x)\,dx, \quad A \in \mathcal{B}([0,1]).$$

Let K be the intersection of all compact subsets E of $[0,1]$ such that $\mu(E) = 1$. Find $\mu(K)$.

3. For a function $f : [0,1] \to \mathbb{R}$ determine whether either of the statements

"f is continuous almost everywhere on $[0,1]$"

and

"there is a continuous $g : [0,1] \to \mathbb{R}$ such that $f = g$ almost everywhere"

implies the other one. In each case justify your answer with a proof or counterexample.

4. Suppose that $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| = \|f\|_{L^1(\mathbb{R})}.$$

REAL ANALYSIS GRADUATE EXAM
Fall 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let μ, ν be finite Borel measures on $\mathbb{R}^2$ such that $\mu(B) = \nu(B)$ for every open triangular region B in the plane. Show that $\mu(E) = \nu(E)$ for all Borel sets E. [Note added later: This is a modified version of the problem actually asked, which was inappropriately difficult, with balls in place of triangles.]

(2) Show that

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n}\, dx$$

exists, and determine its value.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function and let m be Lebesgue measure. Suppose there exist $M > 0$ and $c \in (0,1)$ such that

$$m\big(\{x : |f(x)| \geq t\}\big) \leq \frac{M}{t^c} \quad \text{for all } t > 0.$$

Show that f is Lebesgue integrable.

(4) Let $T_0^1(g) = \sup_{a=x_0<x_1<\cdots<x_n=b} \sum_{i=1}^n |g(x_i) - g(x_{i-1})|$ denote the total variation of a function $g : [0,1] \to \mathbb{R}$. Suppose that f_n, f are real-valued with $f_n(x) \to f(x)$ for all $x \in [0,1]$.

(i) Show that $T_0^1(f) \leq \liminf_{n\to\infty} T_0^1(f_n)$.

(ii) If we also assume each f_n is absolutely continuous and $T_0^1(f_n) \leq 1$ for all n, is it necessarily true that $T_0^1(f) = \lim_{n\to\infty} T_0^1(f_n)$? Justify.

REAL ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1)(a) Let $(X, \mathcal{B}, \mu)$ be a measure space, with μ finite, and let $\mathcal{A} \subset \mathcal{B}$ be an algebra. A set $E \in \mathcal{B}$ is called *approximable from inside by* $\mathcal{A}$ if for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $A \subset E, \mu(E \backslash A) < \epsilon$. Show that $\mathcal{C} = \{E \in \mathcal{B} : E \text{ is approximable from inside by } \mathcal{A}\}$ is closed under countable unions.

(b) Find an example which shows $\mathcal{C}$ need not be closed under complements. HINT: Consider the rationals and irrationals in an interval.

(2) Let f, g be absolutely continuous on $[a, b]$.

(a) Show that fg is absolutely continuous.

(b) Show that the integration by parts formula is valid:

$$\int_{[a,b]} fg' \, dx = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f'g \, dx.$$

(c) Show by example that the integration by parts formula need not be valid if we only assume f, g differentiable a.e. (that is, we do not assume they are absolutely continuous.)

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $f = 0$ outside $[-1, 1]$. Define $f_n(x) = f(x + \frac{1}{n})$. Must $f_n \to f$ in measure? Justify your answer. HINT: What about convergence in L^1? Also, first consider a special subclass of the specified functions f.

(4) Let $\mu(X) < \infty$, and suppose $f \geq 0$ is measurable on X. Prove: f is μ-integrable iff

$$\sum_{n=0}^{\infty} 2^n \mu \left(\{x : f(x) \geq 2^n\}\right) < \infty.$$

REAL ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let f be a bounded function on $\mathbb{R}^n$ and for $\epsilon > 0$ let $M_\epsilon(x) = \sup_{y:|y-x|<\epsilon} f(y)$.
(a) Show that $M(x) = \lim_{\epsilon \to 0} M_\epsilon(x)$ exists for all x.
(b) Show that M is upper semicontinuous, that is, $\limsup_{y \to x} M(y) \leq M(x)$.

(2) Let m be Lebesgue measure on $[0,1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\, dt$. Suppose φ is a Lipshitz function, that is, for some M, $|\varphi(x) - \varphi(y)| \leq M|x-y|$ for all x, y. Show that there exists $g \in L^1(m)$ such that $\varphi(F(x)) = \int_0^x g(t)\, dt$.

(3) Let χ_E denote the indicator function of a set E. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\, dy.$$

Show that f is continuous.

(4) Let m be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant C such that $||f_n - f||_1 \leq \frac{C}{n^2}$ for all $n \geq 1$. Show that $f_n \to f$ a.e. HINT: Consider the sets

$$\{x : |f_n(x) - f(x)| > \epsilon \text{ for some } n \geq N\}.$$

REAL ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c.*) if for all $x \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|y - x| < \delta$.

(i) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x : f(x) < a\}$.

(ii) Suppose μ is a finite measure on $\mathbb{R}$ and A is a closed subset of $\mathbb{R}$. Using (i) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y : y \in A\}$.

(2) Suppose $\{f_n\}$ and f are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$.

In each case, prove or give a counterexample.

(3) Suppose $f : [0, 1] \to \mathbb{R}$ is a strictly increasing absolutely continuous function. Let m denote Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$.

(4) For $n \geq 1$ define h_n on $[0, 1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here χ_E denotes the characteristic function of E. If f is Lebesgue integrable on $[0, 1]$, show that

$$\lim_{n \to \infty} \int_{[0,1]} f h_n \, dm = 0.$$

HINT: First consider f in a suitably smaller function space.

REAL ANALYSIS GRADUATE EXAM
Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

(2)(a) Let m denote Lebesgue measure on $\mathbb{R}$ and let f be an integrable function. Show that for $a > 0$,

$$\int f(ax)\ m(dx) = \frac{1}{a}\int f(x)\ m(dx).$$

HINT: Consider a restricted class of functions f first.

(b) Let F be a measurable function on $\mathbb{R}$ satisfying $|F(x)| \le C|x|$ for all x, and suppose F is differentiable at 0. Show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\ m(dx) = \pi F'(0).$$

HINT: Use (a).

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f be a measurable function with $|f| < 1$. Prove that

$$\lim_{n\to\infty}\int_X (1 + f + \cdots + f^n)\ d\mu$$

exists (it may be ∞.) HINT: First consider $f \ge 0$.

(4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a,b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$ is finite for all $x \in [a,b]$. Show that

$$F'(x) = \sum_{j=1}^{\infty} F_j'(x) \quad \text{for } m\text{-a.e. } x \in [a,b].$$

HINT: Consider the corresponding measures μ_F and μ_{F_j}.

REAL ANALYSIS GRADUATE EXAM
Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $A \subset \mathbb{R}$ and suppose that for each $\epsilon > 0$ there are Lebesgue-measurable sets E, F with $E \subset A \subset F$ and $m(F \backslash E) < \epsilon$. Show that A is Lebesgue measurable.

(2) Let $f > 0$ be a Lebesgue-integrable function on $[0,1]$. Show that

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_{[0,1]} (f^\epsilon - 1)\, dm = \int_{[0,1]} \log f\, dm.$$

Here m denotes Lebesgue measure. HINT: Decompose f (or $\log f$) into two parts.

(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous, and

$$\lim_{h \searrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|\, dx = 0.$$

Show that $f = 0$ a.e.

(4)(a) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, \ldots, F_7$ are 7 measurable sets with $\mu(F_j) \geq 1/2$ for all j. Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \phi$.

(b) Let m denote Lebesgue measure on $[0,1]$, and let $f_n \in L^1(m)$ be nonnegative and measurable with

$$\int_{[0,1/n]} f_n\, dm \geq 1/2$$

for all $n \geq 1$. Show that $\int_{[0,1]} [\sup_n f_n(x)]\; m(dx) = \infty$. HINT: Part (b) does not necessarily use part (a).

REAL ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Find

$$\lim_n \frac{1}{n} \int_0^n x f(x)\, dx.$$

(2) Suppose $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show that f^α is absolutely continuous.

(3)(a) Let $\{\mu_k\}$ be a sequence of finite signed measures. Find a finite positive measure μ such that $\mu_k \ll \mu$ for all k.

(b) Construct an increasing function whose set of discontinuities is $\mathbb{Q}$. (Prove it is a valid example.)

(4) Let m be Lebesgue measure on $\mathbb{R}$. For $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$, define the function $A_r f$ by

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy,$$

which is the average value of f on the ball $B(x,r)$ of radius r centered at x, and define the function Hf by $Hf(x) = \sup_{r>0} A_r|f|(x), x \in \mathbb{R}^d$.

(a) Show that for $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, C', R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for all $|x| > R$ and

$$m\Big(\{x : Hf(x) > \alpha\}\Big) \geq \frac{C'}{\alpha} \quad \text{for all sufficiently small } \alpha.$$

(b) Define the function H^*f by

$$H^*f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|\, dy : B \text{ is a ball containing } x\right\}.$$

Show that $Hf \leq H^*f \leq 2^n Hf$. (Note that unlike Hf, in the definition of H^*f the ball B need not be centered at x.)