Topics for the Graduate Exam in Geometry and Topology

Most of the following topics are normally covered in the courses Math 535a and 540.

This is a two hour exam.

Differentiable manifolds: definition, submanifolds, smooth maps, tangent and cotangent bundles.

Differential forms: exterior algebra, integration, Stokes' theorem, de Rham cohomology.

Lie derivatives: of forms and vector fields.

Differential topology: regular values, Sard's theorem, degree of a map, and index of a vector field.

"Classical" differential geometry: local theory of surfaces, 1st and 2nd fundamental forms, Gauss-Bonnet formula.

Homotopy theory: definition of homotopy, homotopy equivalences, fundamental groups (change of base point, functoriality, Van Kampen theorem, examples such as the fundamental group of the circle), covering spaces (lifting properties, universal cover, regular (or Galois) covers, relation to π_1), higher homotopy groups.

Singular homology theory: definition of the homology groups, functoriality, relative homology, excision, Mayer-Vietoris sequences, reduced homology, connection between H_1 and the fundamental group, homology of classical spaces (e.g. S^n, $R^n - \{0\}$).

References:

M. Berger and B. Gostiaux: Differential Geometry: Manifolds Curves and Surfaces
A. Hatcher, Algebraic Topology
I.M. Singer and J.A. Thorpe: Lecture Notes on Elementary Topology and Geometry
H. Hopf: Differential Geometry in the Large, Springer Lecture Notes in Mathematics, v. 1000
M.J. Greenberg and J.R. Harper: Lectures on Algebraic Topology
J.R. Munkres: Elements of Algebraic Topology
J.W. Vick: Homology Theory
W.S. Massey: Algebraic Topology: An Introduction
I. Madsen and J. Tornehave: From Calculus to Cohomology

GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

SPRING 1993

Problem 1 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable. Compute $f^*(\,dx_1 \wedge \,dx_2)$.

Problem 2 Let $f : M \to N$ be a differentiable map between two manifolds, such that f is bijective and such that its tangent map $T_x f : T_x M \to T_{f(x)} N$ is an isomophism for every $x \in M$. Show that f is a diffeomorphism.

Problem 3 Let $B^2 = \{x \in \mathbb{R}^2; ||x|| \leq 1\}$ be the unit disk in the plane. Let $f : B^2 \to B^2$ be a continuous map such that $f(x) = x$ for every $x \in S^1 = \{x \in \mathbb{R}^2; ||x|| = 1\}$. Show that f is surjective.

Problem 4 Let M be a compact surface in $\mathbb{R}^3$, namely a compact 2-dimensional submanifold of $\mathbb{R}^3$. Show that there is a point $x \in M$ s uch that M lies entirely on one side of the tangent plane $T_x M$.

Problem 5 Is there a covering map $\mathbb{R}^2$-{2 points}$\to \mathbb{R}^2$-{1 point}? (Possible hint: π_1 and H_1).

Problem 6 Let U be an open subset of $\mathbb{R}^n$. Show that U is homeomorphic to no open subset of $\mathbb{R}^p$ with $p < n$. (Possible hint: consider the homology of a pair $(U, U - \{x\})$).

Problem 7 Recall that the tangent bundle TM of a manifold M consists of all pairs $(x, \vec{v})$ where $x \in M$ and $\vec{v}$ is the tangent space $T_x M$ of M at x. Show that TM is an oriented manifold (even when M is not orientable!).

GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

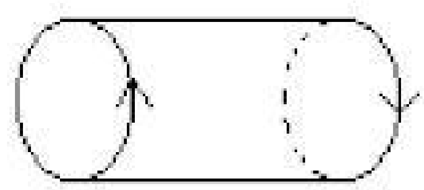
SPRING 1994

Problem 1 Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Does there exist a submersion $f : S^2 \to \mathbb{R}^2$, namely a map such that the tangent map $T_x f : T_x S^2 \to \mathbb{R}^2$ is everywhere surjective?

Problem 2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $M = f^{-1}(0)$. Assume that the tangent map $R_x M : \mathbb{R}^n \to \mathbb{R}$ is non-trivial at each $x \in M$. Is M necessarily a manifold? Is M necessarily orientable? Give a proof or a counterexample.

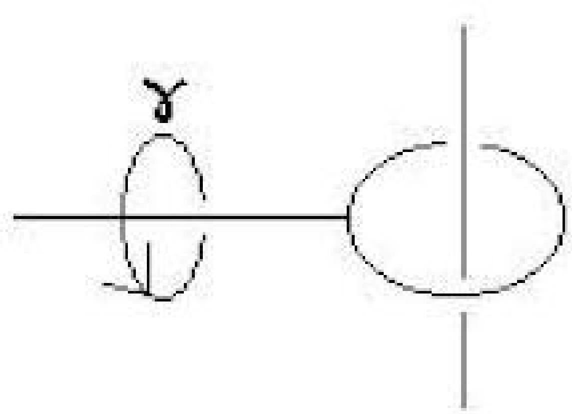
Problem 3 Let $f : \mathbb{C} - \{-1, 0, 1\} \to \mathbb{C} - \{0, 1\}$ be defined by $f(z) = z^2$. Show that the homomorphism $f_* : \pi_1(\mathbb{C} - \{-1, 0, 1\}; 2) \to \pi_1(\mathbb{C} - \{0, 1\}; 4)$ is injective. Compute the groups $\pi_1(\mathbb{C} - \{-1, 0, 1\}; 2)$ and $\pi_1(\mathbb{C} - \{0, 1\}; 4)$ and determine the homomorphism f_*.

Problem 4 Compute the fundamental group of the Klein bottle. (See figure below.)



Problem 5 Let $B_1, \ldots, B_p$ be p disjoint copies of the n-dimensional closed ball B^n, and let X be the space obtained by gluing these balls along their boundary. Namely, choose a homeomorphism $\varphi_i : B^n \to B_i$ for every i. Then, X is the quotient of the space $\cup_{i=1}^p B_i$ by the equivalence relation whose equivalence classes are all $\{x\}$ with x in the interior of some B_i as well as all subsets $\{\varphi_1(y), \varphi_2(y), \ldots, \varphi_p(y)\}$ with $y \in S^n$. Compute the homology groups of X.

Problem 6 Let ω be closed differential form of degree 1 defined on $\mathbb{R}^3 - L$, where L is a subset shown below (made up of the z-axis, the unit circle and a half line in the xy-plane). Let γ be the closed curve shown. Calculate $\int_\gamma i^*(\omega)$, where $i : \gamma \to \mathbb{R}^3 - L$ is the inclusion map. (Hint: Be smart, apply Stokes to a suitably chosen surface).



GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

FALL 1994

Problem 1 Let $X = \mathbb{R}^2 - \{(\frac{1}{n}, 0) | n = 1, 2, \ldots\}$.

(a) Show that the fundamental group $\pi_1(X, (0,0))$ is non-trivial.

(b) Is the fundamental group abelian? Explain.

(c) Is X semi-locally simply connected? Explain.

(d) Does there exist a covering $E \to X$ with E simply connected?

Problem 2 Let M be a manifold of dimension $m \geq 2$ and let $B \subset M$ be an open subset that is homeomorphic to the m-dimensional open ball. Fix $x \in B$ and consider the homoemorphisms

$$H_m(M) \xrightarrow{\alpha} H_m(M, M - \{x\}) \xrightarrow{\beta} H_m(b, B - \{x\}) \xrightarrow{\gamma} H_{m-1}(B - \{x\})$$

where α is induced by the inclusion map $M \to (M, M - \{x\})$, β is the excision isomorphism, and γ is the connecting homomorphism of the long exact sequence in relative homology of the pair $(B, B - \{x\})$. Also, let $H_m(M) \xrightarrow{\delta} H_{m-1}(B - \{x\})$ be the connecting homomorphism of the Mayer-Vietoris exact sequence associated to the decomposition of M as $M = (M - \{x\}) \cup B$. Is δ equal to the composition $\gamma \circ \beta \circ \alpha$?

Problem 3 Let $\boldsymbol{S}^3 \subset \mathbb{R}^4$ be the 3-sphere defined by $w^2 + x^2 + y^2 + z^2 = 1$ where w, x, y, z are the standard Euclidean coordinates on $\mathbb{R}^4$. Let $f : \boldsymbol{S}^3 \hookrightarrow \mathbb{R}^4$ be the inclusion map. Compute the integral of $f * \theta$ over $\boldsymbol{S}^3$, where θ is the 3-form (defined on $\mathbb{R}^4$ minus the origin) given by

$$\theta = \frac{w^7 \, dx \wedge \, dy \wedge \, dz}{w^2 + z^2 + y^2 + z^2}$$

Problem 4 Is the set $X \subset \mathbb{R}^4$ defined by $w^2 + x^2 + y^2 + z^2 = 1$ and $w^2 + x^2 = y^2 + z^2$ a smooth submanifold of $\mathbb{R}^4$?

Problem 5 Can the set $X \subset \mathbb{R}^4$ defined by $w^2+x^2+y^2+z^2 < 1$ and $w^2+x^2 = y^2+z^2$ (considered as a topological subspace of $\mathbb{R}^4$) carry the structure of a smooth manifold?

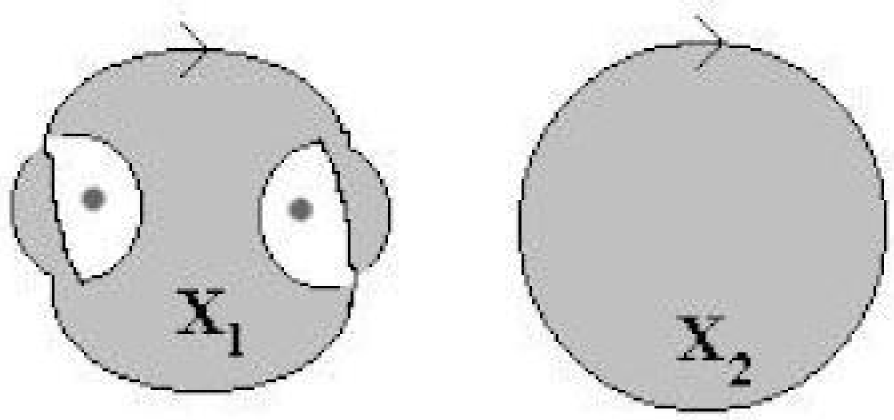
Problem 6 Let $\boldsymbol{S}^n$ be the n-dimensional sphere, and let $\boldsymbol{T}^n = (\boldsymbol{S}^1)^n$ be the n-dimensional torus. Does there exist a submersion from $\boldsymbol{S}^3$ to $\boldsymbol{T}^2$? From $\boldsymbol{T}^2$ to $\boldsymbol{S}^2$? From $\boldsymbol{S}^3$ to $\boldsymbol{S}^2$? (Note: A submersion is a smooth map whose differential at each point is surjective.)

GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

SPRING 1995

Problem 1 Show that the tangent bundle of a differentiable manifold is an oriented manifold.

Problem 2 Let X_1 be the "double Möbius strip" shown below. Let X_2 be the disk. Note that the boundaries ∂X_1 and ∂X_2 of these two surfaces are both homoeomorphic to a circle. Choose a homeomorphism $\varphi : \partial X_1 \to \partial X_2$ and let $X = X_1 \cup X_2 / \sim$, where the equivalence relation $\sim$ identifies each $x_1 \in X_1$ to $\varphi(x_1) \in \partial X_2$. (In other words, X is obtained by gluing X_1 and X_2 along their boundary.) Give a presentation for the fundamental group of X.



Problem 3 Let X be as in Problem 2. Compute all the holomogy groups $H_n(X, \mathbb{Z})$.

Problem 4 Let S^2 be the two-dimensional sphere and let T^2 be the two-dimensional torus. Prove that, for every continuous mapping $f : S^2 \to T^2$, the induced map in homology $H_2(f) : H_2(S^2, \mathbb{R}) \to H_2(T^2, \mathbb{R})$ is zero.

Problem 5 Let $M \subset \mathbb{R}^3$ be the subset defined by $x^6 + y^6 + z^6 = 1$. Prove that M is a smooth submanifold of $\mathbb{R}^3$ and compute the integral of $x^3y^2z^2 \, \mathrm{d}y \wedge \, \mathrm{d}z$ over M.

Problem 6 Two coverings $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ are said to be equivalent if there is a homomorphism $\varphi : \tilde{X} \to \tilde{X}'$ such that $p' \circ \varphi = p$. If X is the figure eight ∞, how many equivalence classes of coverings $p : \tilde{X} \to X$ with $p^{-1}(x) =$(three points) are there?

Problem 7 Use differential geometry to prove the Cauchy Integral Theorem:

If $f : \Omega \to \mathbb{C}$ is a holomorphic function on an open subset Ω of the complex plane, and if $c : [0, 1] \to \Omega$ is a differentiable curve iwth $c(0) = c(1)$ and $[c] = 0$ in $H_1(\Omega, \mathbb{Z})$, then $\int_c f(z) \, \mathrm{d}z = 0$.

Qualifying Exam in Topology and Geometry – Spring 1996

Directions: Do six of the following seven problems. Show clearly all of your work.

Problem 1. (a) State carefully the classification of closed, compact, connected, oriented topological surfaces without boundary. That is, describe a list of such surfaces so that any other such surface is homeomorphic to exactly one surface on your list. Briefly describe the proof. (b) Extend this result to give a classification of closed, compact, connected, oriented topological surfaces with boundary. (c) We say that two simple closed curves C and D in an orientable topological surface M are *equivalent* if there is an orientation-preserving homeomorphism $f : M \to M$ such that $f(C) = D$. Give a complete list of all equivalence classes of curves in a closed, compact, orientable topological surface M without boundary.

Problem 2. Let S^m be the m-dimensional sphere, and let M^m be a smooth compact oriented m-dimensional manifold. Suppose that $f : S^m \to M^m$ is a smooth map of degree one. Prove that M is a cohomology m-spere (i.e. has the same cohomology groups as S^m).

Problem 3. Let $M \subset \mathbf{R}^3$ be a smooth compact surface with constant Gaussian curvature. (a) Explain why M must be diffeomorphic to a two-sphere. (b) Prove that in fact M is a Euclidean two-sphere.

Problem 4. Let $a, b \in \mathbf{C}$ be two points in the complex plane. Assume that

$$\int_a^b z^5 dz = 0 = \int_a^b z^{46} dz.$$

Prove that $a = b$.

Problem 5. Consider the subset $M \subset \mathbf{R}^3$ defined by $x^{30} + y^{30} + z^{30} = 1$. Prove that M is a smooth surface, and compute the integral of $x^{15}y^{14}z^{14}dy \wedge dz$ over M.

Problem 6. Contruct a topological space whose fundamental group is isomorphic to the group $< a, b \mid a^2b^3 = 1 >$.

Problem 7. Let $f(z)$ be a complex polynomial of degree 5. Recall that the extended complex plane $\mathbf{C} \cup \{\infty\}$ can be identified with the two-sphere S^2 via stereographic projection. Think of $f(z)$ as being a map $f : S^2 \to S^2$. Compute the map of homology groups $f_* : H_*(S^2) \to H_*(S^2)$.

Qualifying Exam in Geometry/Topology - May 1997

Problem 1. (a) Define the *degree* of a smooth map between smooth, compact, oriented manifolds of the same dimension. (b) Suppose that M is a smooth, compact, oriented manifold of dimension $m \geq 1$. Prove that there exists a smooth map $f : M \to S^m$ of degree 1. *(Note: Here S^m denotes the standard m-sphere.)*

Problem 2. Using Sard's Theorem together with the classification theorem for one-dimensional manifolds with boundary, prove that there does not exist a smooth retraction from $D^n \to S^{n-1}$. *(Note: Here D^n denotes the closed unit disk and S^{n-1} denotes its boundary, the unit sphere. By a retraction we mean a map $D^n \to S^{n-1}$ whose restriction to $S^{n-1} \subset D^n$ is the identity map.)*

Problem 3. Does there exist a curve segment C in the standard two-sphere $S^2 \subseteq R^3$ running from the South Pole to the North Pole and meeting each latitude (i.e., each level set $z =$ constant) at an angle of $\pi/4$?

Problem 4. Let U be an open subset of $\mathbf{R}^n$, and let $\omega \in \Omega^p(U \times \mathbf{R})$ be a differential form of degree p on $U \times \mathbf{R}$. *(Remark: Here $\Omega^p(X)$ denotes the space of differential forms of degree p on X.)*

1. Show that there exists a family of differential forms $\alpha_t \in \Omega^p(U)$ and $\beta_t \in \Omega^{p-1}(U)$, depending differentiably on t, such that

$$\omega(x,t) = \alpha_t(x) + \beta_t(x) \wedge dt \tag{1}$$

   at every $(x,t) \in U \times \mathbf{R}$.

2. Show that there exists $\omega' \in \Omega^{p-1}(U \times \mathbf{R})$ such that $\omega = d\omega'$ if and only if $d\omega = 0$ and there exists $\alpha' \in \Omega^{p-1}(U)$ such that $\alpha_7 = d\alpha'$.

Problem 5. Compute the homology groups $H_n(S^2 \times S^q; \mathbf{Z})$, where S^p denotes the p-dimensional sphere.

Problem 6. Recall that two covering spaces $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ over the same space X are *isomorphic* if there is a homeomorphism $\phi : \tilde{X} \to \tilde{X}'$ such that $p' \circ \phi = p$. Let X be the Klein bottle $S^1 \times [0,1]/\sim$ where $\sim$ identifies $(z,0)$ to $(\overline{z},1)$. Up to isomorphism, how many coverings $p : \tilde{X} \to X$ with $p^{-1}(\text{ one point }) = \text{ three points}$ are there? *(Hint: Consider monodromy.)*

Qualifying Exam in Geometry/Topology Fall 1997.

1. Let ω be a 1-form defined on the sphere $S^2 = \{x \in R^3 | |x| = 1\}$. Assume ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for any $\phi \in SO(3)$, show $\omega = 0$.

2. Show the set $M = \{x \in R^4 | x_1 x_2 = x_3 x_4, |x| = 1\}$ is a smooth orientable surface.

3. Let M, N be smooth manifolds of dimension n, and $\pi : M \to N$ be a smooth map which is onto and has rank n at each point. Prove or disprove the statements:
  a) π is locally a diffeomorphism;
  b) π is a covering map.

4. Let S^1 be the unit circle in $R^2 = R^2 \times \{0\} \subset R^3$. Compute the fundamental group of $R^3 - S^1$.

5. Compute the homology of $R^3 - S^1$ with coefficients in Z.

6. Let $f : RP^2 \to T^2$ be a continuous map from the projective plane RP^2 to the torus $T^2 = S^1 \times S^1$.
  (a) Show that the induced homomorphism $f_* : \pi_1(RP^2) \to \pi_1(T^2)$ is trivial.
  (b) Show that f is homotopic to a constant map.

Geometry/Topology Exam, Spring 1998

1) Let $X = S^1 \times S^1 - \{x, y\}$, where the two points $x, y \in S^1 \times S^1$ are distinct. Compute the fundamental group $\pi_1(X; x_0)$ and the homology R–modules $H_n(X)$.

2) Let $f : S^n \to S^n$ be a continuous map such that $H_n(f) : H_n(S^n) \to H_n(S^n)$ is non-trivial. Show that f is surjective.

3) Let B^n be the unit ball in $\mathbf{R}^n$, with boundary the $n-1$–sphere S^{n-1}. If $f : B^n \to \mathbf{R}^n$ is a continuous map with $f(S^{n-1}) \subset B^n$ (but not necessarily $f(B^n) \subset B^n$), show that there exists an $x \in B^n$ such that $f(x) = x$.

4) Let $f : M \to M$ be a diffeomorphism of the manifold M so that $f^n = Id$, but $f, f^2, \ldots, f^{n-1}$ have no fixed point. Let M/f denote the quotient space of M by the equivalence relation which identifies $x, y \in M$ when there exists p with $y = f^p(x)$.

a) Show that M/f is a manifold.

b) Compute $\pi_1(M/f)$ if M is simply connected.

5) Consider the map $\phi(x, y, z) = (x^2 - y^2, xy, xz, yz)$.

a) Show that the image of the unit sphere $S^2 \subset \mathbf{R}^3$ (of equation $x^2 + y^2 + z^2 = 1$) is a submanifold of $\mathbf{R}^4$.

b) Show this image $\phi(S)$ is diffeomorphic to the real projective plane $\mathbf{RP}^2$.

6) Calculate the integral

$$\int_{S^2} \omega$$

where S^2 is the standard unit sphere in $\mathbf{R}^3$, and ω is the 2–form

$$\omega = (x^2 + xy)(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy).$$

Geometry/Topology qualifying exam
Spring 1999

1. Let M be an embedded compact surface in $\mathbb{R}^3$, namely a non-empty 2–dimensional submanifold of $\mathbb{R}^3$. Show that there exists an infinite number of vertical lines $L = \{x\} \times \{y\} \times \mathbb{R}$ which meet M and are not tangent to it in the following sense: $M \cap L$ is non-empty and, for every $x \in M \cap L$, the plane tangent to M at x is not vertical.

2. Let N be a n–dimensional submanifold of the m–dimensional manifold M, and let $i : N \to M$ be the inclusion map. Suppose that N is closed in M. Show that, if $\alpha \in \Omega^p(N)$ is a degree p differential form on N, there exists a form $\beta \in \Omega^p(M)$ on M such that $i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β so that $d\beta = 0$?

3. In $B^2 \times B^2$, let X be the union of the torus $S^1 \times S^1$ and of the disk $B^2 \times \{x\}$ (where B^2 is the closed unit disk in $\mathbb{R}^2$ and S^1 is its boundary circle). Compute the fundamental group of X.

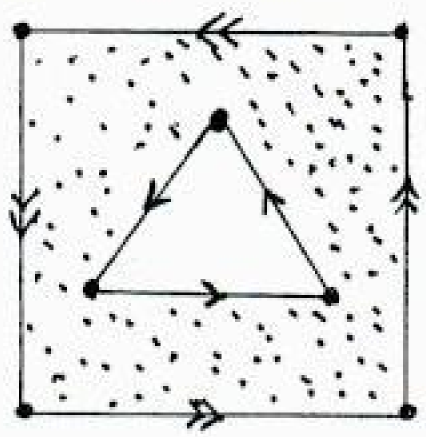
4. For X as in Problem 3, compute the homology modules $H_p(X; R)$, with coefficients in an arbitrary unitary ring R.

5. Prove or disprove: A surjective map $p : \widetilde{X} \to X$ is a covering map if and only if, for every $\widetilde{x} \in \widetilde{X}$, there is a neighborhood $\widetilde{U}$ of $\widetilde{x}$ such that the restriction $p_{|\widetilde{U}} : \widetilde{U} \to p\left(\widetilde{U}\right)$ is a homeomorphism.

6. Let X be a path connected space such that $\pi_1(X; x_0) = 1$ and $\pi_2(X; x_0) = 1$. Recall that the second property means that, for every continuous map $\alpha : [0,1] \times [0,1] \to X$ such that $\alpha(s,t) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$, there is a homotopy $H : [0,1] \times [0,1] \times [0,1] \to X$ such that $H(s,t,u) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$ or $u = 1$. Consider the 2–dimensional torus $T^2 = S^1 \times S^1$. Show that every continuous map $f : T^2 \to X$ is homotopic to a constant map. (Possible hint: Write the torus as a square with identifications of its sides.)

Geometry/Topology Graduate Exam
Fall 1999

1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



2. Let X be the space described in 1. Compute the homology groups $H_n(X;\mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.

3. Give an example of a path connected space X which admits no covering $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected.

4. Let X be a path connected manifold with $\pi_1(X;x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

5. You may know that there exist continuous surjective maps $f : [0,1] \to [0,1]^2$ from the interval onto the square. Show that there exists no *continuously differentiable* surjective map $f : [0,1] \to [0,1]^2$.

6. Consider the map $\varphi : S^1 \times S^1 \to S^1 \times S^1$ defined by $\varphi(u,v) = (u^5, v^{-3})$, where we identify S^1 to the unit circle in the complex plane $\mathbb{C}$. Compute the degree of φ.

7. Let $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ be a closed (namely $d\omega = 0$) differential form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Consider the homomorphism $i^* : \Omega^n(\mathbb{R}^{n+1} - \{0\}) \to \Omega^n(S^n)$ induced by the inclusion map $i : S^n \to \mathbb{R}^{n+1} - \{0\}$. Show that the form ω is exact (namely there exists $\alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ such that $\omega = d\alpha$) if and only if $\int_{S^n} i^*(\omega) = 0$.

Qualifying Exam in Geometry/Topology Fall 2000

1. Let ω be a 1-form defined on the sphere $S^2 = \{x \in R^3 || x| = 1\}$. Assume ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for any $\phi \in SO(3)$, show $\omega = 0$.

2. Show the set $M = \{x \in R^4 | x_1 x_2 = x_3 x_4, |x| = 1\}$ is a smooth orientable surface.

3. Let M, N be smooth manifolds of dimension n, and $\pi : M \to N$ be a smooth map which is onto and has rank n at each point. Prove or disprove the statements:

a) π is locally a diffeomorphism;

b) π is a covering map.

4. Let S^1 be the unit circle in $R^2 = R^2 \times \{0\} \subset R^3$. Compute the fundamental group of $R^3 - S^1$.

5. Compute the homology of $R^3 - S^1$ with coefficients in Z.

6. Let $f : RP^2 \to T^2$ be a continuous map from the projective plane RP^2 to the torus $T^2 = S^1 \times S^1$.

(a) Show that the induced homomorphism $f_* : \pi_1(RP^2) \to \pi_1(T^2)$ is trivial.

(b) Show that f is homotopic to a constant map.

Graduate Exam in Topology/Geometry

February 2002

1. Let $P^n(\mathbb{R})$ be the projective n–space, namely the quotient space of the sphere S^n by the equivalence relation $\sim$ defined by $x \sim y \Leftrightarrow x = \pm y$.

(a) Show that $P^n(\mathbb{R})$ is a manifold.

(b) Show that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

2. In the set $M(n)$ of all $n \times n$ matrices, identified to $\mathbb{R}^{n^2}$, consider the subset $O(n)$ consisting of the orthogonal matrices, namely those matrices A for which AA^t is the identity (where A^t denotes the transpose). Show that $O(n)$ is a submanifold of $M(n) = \mathbb{R}^{n^2}$, and that the tangent space $T_{\mathrm{Id}}O(n)$ at the identity Id is equal to the space of all antisymmetric matrices (namely those matrices for which $A^t = -A$).

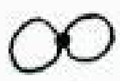
3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (\alpha x + \beta y, \gamma x + \delta y, \varepsilon z)$, where α, β, γ, δ, ε are constants with $\alpha\delta - \beta\gamma = 1$. Find the matrix of $f^* : \wedge^2\mathbb{R}^3 \to \wedge^2\mathbb{R}^3$ associated to the basis $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$.

4. Let $P^2(\mathbb{R})$ be the real projective plane.

(a) If $x \in P^2(\mathbb{R})$, compute the fundamental group $\pi_1(P^2(\mathbb{R}) - \{x\})$.

(b) Show that any map $f : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which is not surjective is homotopic to a constant map. (Hint: use a covering space).

5. Let B^2 be the closed 2–dimensional ball, with boundary the circle S^1. Let $X = S^1 \times B^2$ and let $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_n(X, \partial X)$ with coefficients in Z. (You are allowed to use whatever you may know about the homology of the torus ∂X).

6. Let X be the figure eight , union of two circles C_1 and C_2 meeting in one point. Let $p : \widetilde{X} \to X$ be a covering space such that $\widetilde{X}$ is connected and such that the preimage $p^{-1}(x)$ of each $x \in X$ consists of 2 points. Compute the fundamental group of $\widetilde{X}$.

7. What are the compact connected surfaces S for which there exists an immersion $S \to S$ which is not a diffeomorphism? (Hint: Euler characteristic).

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Geometry/Topology Qualifying Exam

February 2003

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold M of dimension $2n$ (without boundary), and let ω be a *symplectic form* on M, namely a differential form of degree 2 whose n–th exterior power $\omega \wedge \omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove that the second de Rham cohomology $H^2_{dR}(M; \mathbf{R}) \neq 0$ by showing that ω is not exact.

2. Show that the set $Sl(n, \mathbf{R})$ of $n \times n$ matrices A with entries in the real numbers and which satisfy $\det(A) = 1$ is a manifold. What is its dimension?

3. On $\mathbf{R}^4$ with coordinates x_1, y_1, x_2, y_2, consider the 2-form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a smooth function f on $\mathbf{R}^4$, let X be the vector field
$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}.$$
Then compute $\mathcal{L}_X\omega$, the Lie derivative of ω in the direction X.

4. Let M be a compact oriented n-dimensional manifold (without boundary), where $n > 1$. Show that there exists a differentiable map $f : M \to S^n$ of degree 1.

5. Recall that two coverings $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *equivalent* if there exists a homeomorphism $\varphi : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. When X is the 2-dimensional torus $S^1 \times S^1$, determine the number of equivalence classes of all coverings $p : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 3 points (for an arbitrary x_0).

6. Compute the homology groups $H_n(X; \mathbf{Z})$ of the complement $X = \mathbf{R}^5 - A$ of a subset $A \subset \mathbf{R}^5$ consisting of 4 points.

7. Let B^n be the closed unit ball in $\mathbf{R}^n$, and let S^{n-1} be its boundary, namely the $(n-1)$–dimensional sphere. If $f : B^n \to \mathbf{R}^n$ is a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$, show that the image $f(B^n)$ contains the ball B^n.

Geometry/Topology Qualifying Exam
Fall 2003

1. Let T^n be the n–dimensional torus $S^1 \times S^1 \times \cdots \times S^1$. Construct a differentiable embedding of T^n in $\mathbf{R}^{n+1}$.

2. Let S^n denote the n–dimensional sphere, and consider a differentiable map $f : S^n \to \mathbf{R}^n$ such that $f(S^n)$ has non-empty interior in $\mathbf{R}^n$.

a) Warm-up: Show there there is at least one point $x \in S^n$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f_{|U} : U \to f(U)$ is a diffeomorphism. .

b) Show that there exists at least two points x, $y \in S^n$ such that f is a local diffeomorphism at x and y, f is orientation-preserving at x, and f is orientation-reversing at y.

3. Let M be a manifold with fundamental group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/3) \times (\mathbb{Z}/5)$. Up to isomorphism, how many 3–fold covers does it have? Recall that a 3–fold cover is a covering map $p : \widetilde{M} \to M$ such that each $p^{-1}(x)$ consists of 3 points, and that two such covers $p : \widetilde{M} \to M$ and $p' : \widetilde{M}' \to M$ are isomorphic if there exists a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}'$ such that $p' \circ \varphi = p$.

4. Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$–dimensional submanifold N of M. Show that $d\omega = 0$. (hint: look at small spheres.)

5. Let S^n denote the n-dimensional sphere and define $X = S^1 \times S^2$. Also, choose a point $p_n \in S^n$, for $n = 1, 2, 3$, and take the quotient Y of the disjoint union of S^1, S^2, S^3 by the equivalence relation identifying p_1, p_2, p_3 to a single point $p \in Y$.

a) Calculate the homology groups of X and of Y.

b) Calculate the fundamental groups as well.

c) Are these spaces homeomorphic?

6. Let $T = S^1 \times S^1$ denote the 2-dimensional torus. Identify the circle S^1 to $\{z \in \mathbb{C}; |z| = 1\}$, and the 2–dimensional disk B^2 to $\{z \in \mathbb{C}; |z| \leq 1\}$ in the complex plane $\mathbb{C}$. Adjoin to T two copies D_1 and D_2 of B^2, where the boundary $\partial D_1 = \partial B^2$ of the disk D_1 is glued to $S^1 \times \{1\}$ by the map $z \mapsto z^3$ and where the boundary ∂D_2 of D_1 is glued to $\{1\} \times S^1$ by the map $z \mapsto z^5$. Calculate the fundamental group of X.

Geometry/Topology Qualifying Exam

February 2004

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold of dimension n (without boundary). Let $\omega \in \Omega^n(M)$ be an n-form on M and X a vector field on M. Prove that $\mathcal{L}_X\omega = 0$ at some point $p \in M$. (Here $\mathcal{L}_X\omega$ is the Lie derivative of ω in the direction X.)

2. Let
$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
be a 2-form defined on $\mathbf{R}^3 - \{0\}$. If $i : S^2 = \{x^2+y^2+z^2 = 1\} \to \mathbf{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbf{R}^3$ maps $(x,y,z) \to (3x, 2y, 8z)$.

3. Consider the set $X \subset \mathbf{R}^4$ defined by the simultaneous equations $x^2+y^2-z^2-w^2 = 1$ and $xz + yw = 1$. Is X a smooth submanifold of $\mathbf{R}^4$?
4. Show that any smooth function $g : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point. Here $\mathbf{RP}^k$ is the *real projective space*, defined as the quotient of the k-dimensional sphere $S^k = \{|x| = 1\} \subset \mathbf{R}^{k+1}$ by the equivalence relation $x \sim -x$.
5. Let $S^1 = \{x^2+y^2 = 1, z = 0\}$ denote the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$ (where $\mathbf{R}^3$ has standard coordinates (x,y,z)). Calculate the fundamental group of $\mathbf{R}^3 - S^1$.
6. Let X be a connected covering space of the 2-dimensional torus $T^2 = S^1 \times S^1$. List all the possible homeomorphism types of X.
7. For a topological space X, its *suspension* ΣX is the quotient $(X \times [0,1])/\sim$ of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. (More precisely, the equivalence relation $\sim$ is given by:
$$\forall x, x' \in X \ (x,0) \sim (x',0) \text{ and } (x,1) \sim (x',1).)$$
For any $p \geq 2$, prove that $H_p(\Sigma X, \mathbf{Z})$ is isomorphic to $H_{p-1}(X, \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers. What happens when $p = 0, 1$?

Geometry/Topology Qualifying Exam

September 2004

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Prove that a k-form ω on a k-dimensional torus T^k is exact if and only if $\int_{T^k} \omega = 0$.

2. Consider the following $(n-1)$-form ω on $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$:

$$\omega = \frac{\sum_{i=1}^n (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_n}{(\sum_{i=1}^n x_i^2)^{n/2}},$$

where $\widehat{dx_i}$ means the dx_i term is omitted.
(a) Show that the form ω is closed on $\mathbf{R}^n - \{0\}$.
(b) Compute $\int_E \omega$, where E is the ellipsoid

$$E = \left\{ \frac{x_1^2}{9} + \sum_{i=2}^n x_i^2 = 2004 \right\},$$

and the orientation of E is the outward orientation (induced from the compact region of $\mathbf{R}^n$ bounded by E). You may leave your answer in terms of the volume $vol(B^n)$ of the n-dimensional unit ball B^n.

3. Let X be the topological space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ on the torus. [Here a *Möbius band* is obtained from $[0,1] \times [0,1]$ by identifying $(x, 0) \sim (1-x, 1)$ for all $x \in [0,1]$.]
(a) Compute its fundamental group $\pi_1(X)$.
(b) Compute its homology groups $H_n(X; \mathbf{Z})$ for all $n \geq 0$.

4. Carefully state the Gauss-Bonnet Theorem and use it compute the total Gaussian curvature $\int_\Sigma \kappa$, where Σ is a compact oriented surface of genus 2004 which is embedded in $\mathbf{R}^3$.

5. Let X be the topological space obtained from $\mathbf{R}^3$ (with standard coordinates (x, y, z)) by removing two subsets $A_1 = \{x = y = 0\}$ (the z-axis) and $A_2 = \{x^2+y^2 = 1, z = 0\}$ (the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$). Calculate the fundamental group of X.

6. Show that there exists no smooth (C^∞-differentiable) surjective map from S^2 to S^3.

Continued on the next page.

7. Let f be a homogeneous polynomial in k (real) variables. Homogeneity means that there is some positive integer m for which

$$f(tx_1, \ldots, tx_k) = t^m f(x_1, \ldots, x_k),$$

for all $t \in \mathbf{R}$ and $x_1, \ldots, x_k \in \mathbf{R}$. Prove that the set of points $x \in \mathbf{R}^k$ for which $f(x) = a$ is a $(k-1)$-dimensional submanifold of $\mathbf{R}^k$, provided $a \neq 0$. [Hint: Use Euler's identity for homogeneous polynomials, which states that $\sum_{i=1}^k x_i \frac{\partial f}{\partial x_i} = m \cdot f$.]

Geometry/Topology Qualifying Exam

February 2005

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. For each $n > 0$ and every $m \in \mathbf{Z}$, show that there exists a smooth map $f : S^n \to S^n$ of degree m.

2. Let $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ be a 2-dimensional torus with standard Euclidean coordinates (x, y) inherited from $\mathbf{R}^2$.
   (a) Prove that for any 2-form ω_2 on T^2 there is a 1-form ω_1 on T^2 and a real number a such that
   $$\omega_2 = a dx \wedge dy + d\omega_1.$$
   (b) Prove that for any closed 1-form ω_1 on T^2 there is a smooth function f on T^2 and real numbers a, b so that
   $$\omega_1 = a dx + b dy + df.$$

3. Let M be a nonorientable smooth manifold and $i : M \to \mathbf{R}^m$ be an immersion. Define the *normal bundle* $\nu \to M$ to be the set of points (x, v) where $x \in M$ and $v \in \mathbf{R}^m$ is orthogonal to $i_*(T_x M)$ (with respect to the standard Euclidean metric on $\mathbf{R}^m$). Here i_* is the induced map $T_x M \to T_{i(x)}\mathbf{R}^m$ between tangent spaces and we are identifying $T_{i(x)}\mathbf{R}^m$ with $\mathbf{R}^m$.
   (a) Prove that ν can be given the structure of a smooth manifold.
   (b) Is ν an orientable manifold?

4. Let A be a nonsingular symmetric $n \times n$ matrix and c a nonzero real number. (A matrix is *nonsingular* if $\det A \neq 0$ and *symmetric* if $A^T = A$.) Show that
   $$\{x \in \mathbf{R}^n \mid \langle x, Ax \rangle = c\}$$
   is a submanifold of $\mathbf{R}^n$. Here $\langle , \rangle$ is the standard inner product on $\mathbf{R}^n$. What is the dimension of the submanifold?

5. Compute the second homotopy group $\pi_2(S^2 \vee S^1)$ of the wedge sum of S^2 and S^1.

6. Let Σ be an embedded compact surface without boundary in $\mathbf{R}^3$. Then prove that there is a point $x \in \Sigma$ where the Gaussian curvature $K(x)$ is positive. Here the Gaussian curvature is computed with respect to the metric induced from $\mathbf{R}^3$.

Continued on the next page.

7. Let X be the complement of the knot K in the solid torus $S^1 \times D^2$ as in Figure 1. Compute the homology groups $H_i(X; \mathbf{Z})$.

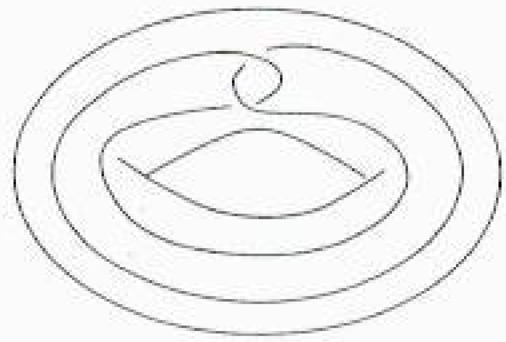


FIGURE 1

Qualifying Exam in Geometry/Topology Fall 2005

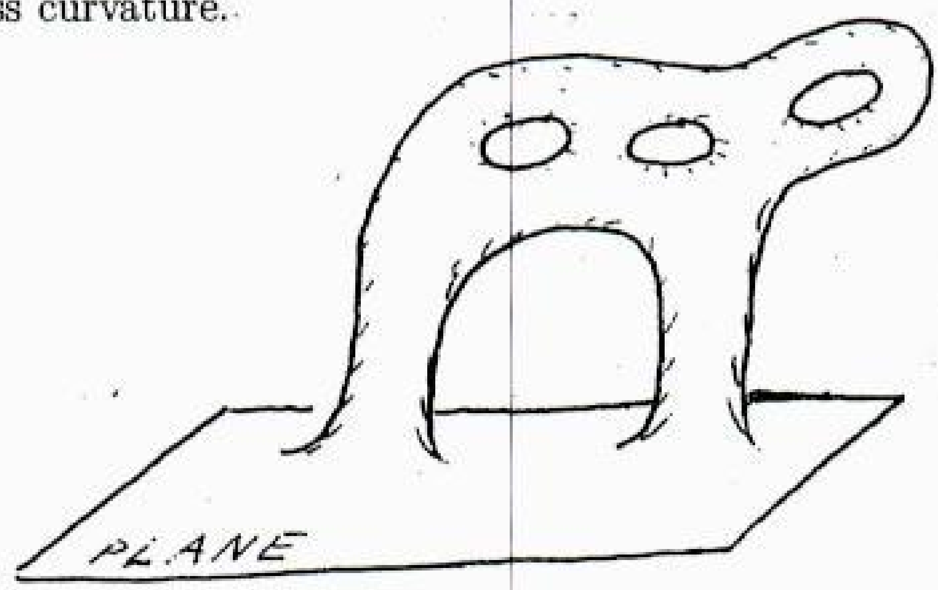
Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Show that the complement of a finite set of points in R^n is simply connected if $n \geq 3$.

2. Fix a space X and say that two covers $p_i : \tilde{X}_i \to X$, for $i = 1, 2$, are *equivalent* if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$. Recall that real projective 2-space RP^2 has its fundamental group isomorphic to the integers mod two, and describe the equivalence classes of connected covers of $RP^2 \times RP^2$.

3. Let α be a closed 2-form on $S^4 = \{(x_1, \ldots, x_5) \in R^5 : x_1^2 + \cdots + x_5^2 = 1\}$. Show that $\alpha \wedge \alpha = 0$ at some point $p \in S^4$.

4. Consider the surface $M \subset R^3$ pictured below. Compute the integral $\int_M K dA$, where K is the Gauss curvature.



(picture of the surface M)

5. Show that for any space X, we have $H_i(X \times S^1) \approx H_i(X) \oplus H_{i-1}(X)$, where S^1 denotes the circle.

6. Given a smooth manifold M, define the *cotangent bundle* $T^*(M)$ to be the set of all pairs (p, q), where $p \in M$ and q lies in the dual vector space to the tangent space $T_p(M)$ of M at p. Show that $T^*(M)$ has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

7. Let M be a smooth manifold. Let $\Omega_c^i(M) \subset \Omega^i(M)$ be the set of smooth i-forms with compact support, i.e., $\omega \in \Omega_c^i(M)$ is zero outside a compact set. Then there is a chain complex

$$0 \to \Omega_c^0(M) \xrightarrow{d_0} \Omega_c^1(M) \xrightarrow{d_1} \Omega_c^2(M) \xrightarrow{d_2} \cdots,$$

where d is the exterior derivative restricted to forms with compact support. Define the ith de Rham cohomology of M *with compact support* to be $ker(d_i)/im(d_{i-1})$. Compute the ith de Rham cohomology of the real line R with compact support for all $i \geq 0$. (Your answer will differ from the usual de Rham cohomology of R.)

Geometry/Topology Qualifying, Spring 2006

Partial credit for partial solutions

1. Let (x, y, z, w) be Cartesian coordinates on $\mathbf{R}^4$. Is the set defined by the equation $x^2 + xy^3 + yz^4 - w^5 = -1$ a smooth manifold of $\mathbf{R}^4$? Prove your assertion.

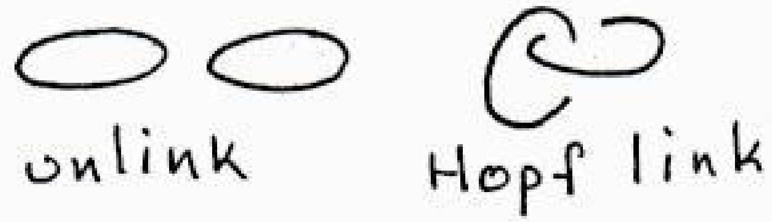
2. **a)** State the definition of the ith de Rham cohomology group $H^i_{dR}(M)$ of a smooth manifold M.
b) Compute the ith de Rham cohomology groups of the real line $\mathbf{R}$ directly from the definition for all $i \geq 0$.

3. Let X be the quotient space obtained from the n-dimensional sphere S^n by identifying three distinct points to a single common point $p \in X$. In other words, let $q, r, s \in S^3$ be pairwise distinct points, let $X = S^n/\sim$ where $x \sim y$ if $x = y$ or if $x, y \in \{q, r, s\}$, and let $p \in X$ denote the equivalence class $\{q, r, s\}$. Calculate $\pi_1(X, p)$.

4. Let $S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbf{R}^4$ and let $\omega = w\ dx \wedge dy \wedge dz$. Compute $\int_{S^3} \omega$.

5. Recall that the *genus* of a closed orientable surface Σ is defined to be $\frac{1}{2} \dim_{\mathbf{R}} H^1_{dR}(\Sigma)$. Let S and T be closed orientable surfaces of respective genera $g(S)$ and $g(T)$. Assume $g(S) < g(T)$. Show that the degree of any smooth map $h : S \to T$ equals zero. [You may use the fact that on a closed orientable surface Σ, the wedge product of one-forms induces a skew-symmetric non-degenerate bilinear pairing $H^1_{dR}(\Sigma) \otimes H^1_{dR}(\Sigma) \to H^2_{dR}(\Sigma) \approx \mathbf{R}$, where $H^i_{dR}(F)$ denotes the ith de Rham cohomology group of Σ.]

6. Define the *unlink* to be the union of two unknotted circles in the three-dimensional sphere S^3, where there are two disjoint three-dimensional balls in S^3 containing the circles. Define the *Hopf link* to be the union of two unknotted disjoint circles in S^3, where each circle meets a disk bounding the other circle in a single point. These links are illustrated in the figure below drawn in $\mathbf{R}^3 = S^3 - \{\text{the point at infinity}\}$. Let U be the complement in S^3 to the unlink and let H be the complement in S^3 to the Hopf link. Calculate the homology groups of U and H.



7. Let X denote a bouquet of $n + 1$ circles, i.e., X is the quotient of the disjoint union of $n + 1$ circles with base points obtained by identifying all the base points to a single point p in the quotient.
a) Prove that $\pi_1(X, p)$ is a free group F_{n+1} on $n + 1$ generators.
b) Let H be a subgroup of F_{n+1} of index k. Show that H is a free group with $kn + 1$ generators.

Geometry/Topology Qualifying Exam

September 2006

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Let M, N be compact oriented manifolds of dimension n (without boundary), and let $f : M \to N$ be a differentiable map. Prove that, if the induced homomorphism $f^* : H^n_{dR}(N;\mathbb{R}) \to H^n_{dR}(M;\mathbb{R})$ between de Rham cohomology groups is surjective, then f is surjective.

2. Let D^2 be the closed unit disk in the complex plane $\mathbb{C}$, bounded by the unit circle S^1. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$ and two copies D_1 and D_2 of D^2. For two integers p, q, let X_{pq} be the quotient space of the disjoint union
$$T^2 \sqcup D_1 \sqcup D_2$$
by the equivalence relation that identifies each point $e^{i\theta}$ in the boundary of D_1 to $(e^{ip\theta}, 1) \in S^1 \times S^1$, and identifies each point $e^{i\phi}$ in the boundary of D_2 to $(1, e^{iq\phi}) \in S^1 \times S^1$. Compute the fundamental group of X_{pq}.

3. Prove that any two continuous maps $f, g : X \to S^1$ from a simply-connected space X to the circle S^1 are homotopic.

4. Calculate the relative homology groups $H_*(S^1 \times D^2, S^1 \times \partial D^2)$, where D^2 denotes the 2-dimensional closed disk and S^1 is the circle.

5. Let M be a compact oriented n-manifold with $H^1_{dR}(M;\mathbb{R}) = 0$ and let $f : M \to T^n$ be a smooth map. Show that the degree of f is equal to 0. (Possible hint: Write $T^n = S^1 \times \cdots \times S^1$; if θ_i is the angular coordinate for the i-th factor S^1, then $d\theta_1 \wedge \cdots \wedge d\theta_n$ is a volume form for T^n.)

6. Recall that the *rank* of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

7. Consider the group SO(3) of orientation-preserving isometries of the 2-dimensional sphere S^2. Namely, SO(3) consists of all rotations of $\mathbb{R}^3$ whose axis passes through the origin or, equivalently of all 3×3 matrices A such that $AA^t = \mathrm{Id}$ and $\det(A) = 1$. Prove that, if ω is a 1-form (not necessarily closed) on S^2 such that $\phi^*(\omega) = \omega$ for every $\phi \in \mathrm{SO}(3)$, then $\omega = 0$.

Geometry/Topology Qualifying Exam

Spring 2007

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (15 pts) Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. (This is, of course, a differentiable manifold.) For $A \in M_n(\mathbb{R})$, define a tangent vector to $M_n(\mathbb{R})$ at the identity matrix I to be the class of the curve $I + tA$, $-\epsilon < t < \epsilon$. Denote this tangent vector by $\vec{A}$.
   (a) For any $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $R_X(B) = B \cdot X$. Prove that R_X is differentiable.
   (b) For any $\vec{A} \in T_I M_n(\mathbb{R})$, define a vector field $\xi_{\vec{A}}$ on $M_n(\mathbb{R})$ so that $\xi_{\vec{A}}(X) = (R_X)_*(I)(\vec{A})$ (Here $(R_X)_*(I)$ is the derivative of R_X at I.) Compute the Lie bracket $[\xi_{\vec{A}}, \xi_{\vec{B}}]$.

2. (15 pts) Let C be the subset of $\mathbb{C}^2$ with coordinates z, w, defined by the equation $w^2 = P(z)$, where $P(z)$ is a polynomial of degree 3.
   (a) Prove that if P has no repeated roots, then C is a submanifold of $\mathbb{C}^2$. (Remark: C is a complex submanifold, and hence is also a real submanifold.)
   (b) Suppose that P has no repeated roots. Compute the fundamental group of $C - \{(z, w) | w = 0\}$. (Hint: Think of covering spaces.)

3. (10 pts) Prove that the tangent bundle TM of a smooth manifold M has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

4. (10 pts) Consider the differential 1-form $\omega = dz - y\,dx$ on $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

5. (10 pts) Define the notion of a *deformation retraction* of a space X onto a subset $A \subset X$. Prove that if A is the knot in the solid torus $X = S^1 \times D^2$ as drawn in the picture below, then there is no deformation retraction of X onto A.

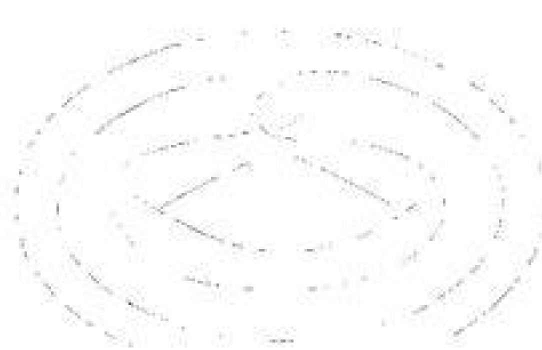


FIGURE 1

6. (10 pts) Construct a topological space X such that $H_0(X;\mathbb{Z}) = \mathbb{Z}$, $H_3(X;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$, $H_5(X;\mathbb{Z}) = \mathbb{Z}$, and all other homology groups are zero.

Graduate Exam
Geometry and Topology
Fall 2007

Problem 1. Let X be a path connected space such that $H_p(X, \mathbb{Z}) = 0$ for every p with $0 < p \leq n$. If $X \times S^n$ denotes the product of X with the n–dimensional sphere S^n, compute the homology groups $H_p(X \times S^n; \mathbb{Z})$ for every p with $0 < p \leq n$.

Problem 2. Let C_1 and C_2 be two disjoint circles in $\mathbb{R}^3$, and let $A = S^1 \times [0, 1]$ denote the cylinder. Let X be the space obtained from the disjoint union $X \sqcup A$ by gluing the boundary component $S^1 \times \{0\}$ of A to the circle C_1 by a homeomorphism, and by gluing the other boundary component $S^1 \times \{1\}$ to C_2 by another homeomorphism. Compute the fundamental group of the space X so obtained.

Problem 3. Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices with coefficients in $\mathbb{R}$, and consider the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$, which to a matrix A associates its determinant $det(A)$. Compute the differential map (also called tangent map) of the function det at the identity matrix $I_n \in M_n(\mathbb{R})$.

Problem 4. Let M be a compact orientable n–dimensional manifold whose boundary ∂M is homeomorphism to the sphere $S^{n-1} \subset \mathbb{R}^n$ by a homeomorphism $f : \partial M \to S^{n-1}$. Let F be a continuous map $F : M \to \mathbb{R}^n$ whose restriction to the boundary ∂M coincides with f. Show that the image $F(M)$ necessarily contains the center O of the sphere S^{n-1}.

Problem 5. Let Ω be the open shell in $\mathbb{R}^2$ consisting of those $(x, y) \in \mathbb{R}^2$ such that $1 < x^2 + y^2 < 10$, and consider the 1–form

$$\omega = \frac{x\,dy - y\,dx}{4x^2 + y^2}$$

a) Show that ω is closed in Ω.

b) Show that ω is not closed in Ω. (Possible hint: consider an ellipse of equation $4x^2 + y^2 = \text{constant}$).

Problem 6. Let $\mathbb{RP}^2$ denote the real projective plane of dimension 2. Consider the map $\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ which to $(x, y) \in \mathbb{R}^2$ associates the element of $\mathbb{RP}^2$ represented by the line passing through the point $(x, y, 1)$. (Recall that $\mathbb{RP}^2$ is the space of lines passing through the origin in $\mathbb{R}^3$.) If $C = \{(x, y) \in \mathbb{R}^2; y^2 = x^3 - x\}$, show that the closure $\overline{\varphi(C)}$ of $\varphi(C)$ in $\mathbb{RP}^2$ is a differentiable submanifold of $\mathbb{RP}^2$.

Problem 7. Let M and N be two compact connected manifolds of the same dimension n, and let $f : M \to N$ be a continuous map. Suppose that the homomorphism $H_n(f) : H_n(M; \mathbb{Z}) \to H_n(\mathbb{Z})$ induced by f is not 0. If $f_* : \pi_1(M, x_0) \to \pi_1(N, f(x_0))$ is the homomorphism induced by f between the fundamental groups, show that its image $f_*(\pi(M; x_0))$ has finite index in $\pi(N; f(x_0))$. (Possible hint: Consider a suitable covering of N.)

Geometry and Topology Graduate Exam
February 2008

1. Let $p : \widetilde{X} \to X$ be a covering with path connected base X, and let G be its automorphism group, consisting of those homeomorphisms $\varphi : \widetilde{X} \to \widetilde{X}$ such that $p \circ \varphi = \varphi$. Pick base points $x_0 \in X$ and $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. Suppose that, for any two $\widetilde{x}_0', \widetilde{x}_0'' \in p^{-1}(x_0)$, there exists $\varphi \in G$ such that $\varphi(\widetilde{x}_0'') = \widetilde{x}_0'$. Show that there is an exact sequence

$$1 \longrightarrow \pi_1(\widetilde{X}; \widetilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \longrightarrow G \longrightarrow 1.$$

2. Consider on $\mathbb{R}^n$ the standard inner product $(\vec{a}, \vec{b}) = \sum_{i=1}^n a_i b_i$, when $\vec{a} = (a_1, a_2, \ldots a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$. Let V be a vector subspace of $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to V$ be the orthogonal projection with respect to the above inner product. If M is a submanifold of $\mathbb{R}^n$, show that the restriction $\pi_{|M} : M \to V$ is an immersion if and only if $T_x M \cap V^\perp = \{0\}$ for every $x \in M$.

3. Let $f : X \to X$ be a map homotopic to a constant map, and let $M_f = X \times [0,1]/\sim$ where the equivalence relation $\sim$ identifies $(x, 0)$ to $(f(x), 1)$. Compute the homology groups of M_f.

4. Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f.

a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α. It may be useful to remember that the map $H^n(S^n) \to \mathbb{R}$ defined by $\gamma \mapsto \int_{S^n} \gamma$ is an isomorphism.

5. Let $\omega \in \Omega^2(S^2)$ be the restriction of the 2–form

$$x\, dy \wedge dz + z\, dx \wedge dy + y\, dz \wedge dx$$

to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. Compute the integral $\int_{S^2} \omega$.

6. Recall that the 1–dimensional projective space $\mathbb{RP}^1$ consists of all lines in $\mathbb{R}^2$ passing through the origin. Let $f : \mathbb{R} \to \mathbb{RP}^1$ associate to $x \in \mathbb{R}$ the line passing through $(x, 1)$ and the origin. Finally, let $P(x)$ be a polynomial function of the variable x.

a) Show that there is no differential form ω on $\mathbb{RP}^1$ such that $f^*(\omega) = P(x)\, dx$.

b) Show that there exists a vector field V on $\mathbb{RP}^1$ such that $f^*(V) = P(x) \frac{\partial}{\partial x}$ if and only if the degree of $P(x)$ is ≤ 2.

7. Let M be a compact differentiable manifold, and let $C^\infty(M)$ be the algebra of all differentiable functions $M \to \mathbb{R}$. Let $\mathcal{I}$ be a maximal ideal of $C^\infty(M)$. Show that there is a point $x_0 \in M$ such that $\mathcal{I} = \{f \in C^\infty(M); f(x_0) = 0\}$. (Possible hint: Suppose that the property is not true and show that, for every $x \in M$, there exists a non-negative function $f \in \mathcal{I}$ such that $f(x) > 0$.)

Geometry/Topology Qualifying Exam

Fall 2008

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.
   (a) Show that d_f is a cochain map, i.e., $d_f \circ d_f = 0$.
   (b) Let $H^i_f(M)$ be the ith cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^0_f(M) \cong \mathbb{R}$ when M is the real line $\mathbb{R}$.
2. Show that, when $m, n > 0$, the homomorphism $f^* : H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$ induced in de Rham cohomology by $f : S^{m+n} \to S^m \times S^n$ is trivial for all $k > 0$. Here S^n is the n-dimensional sphere. [Possible hint: Construct a volume form for $S^m \times S^n$ from a volume form on S^m and a volume form on S^n.]
3. Prove that the set $C = \{(x, y) \mid y^2 - x^3 = 0\}$ is not a smooth submanifold of the plane. [Hint: What is the space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ which are tangent to C?]
4. Let T be the surface obtained by revolving the circle $\{(x, y, z) \mid z = 0, (x - R)^2 + y^2 = r^2\}$ around the y-axis, where $R > r$. Compute the integral
$$\int_T x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$
5. Let B^3 be the (closed) 3-dimensional ball, and let K be a closed, connected 1-dimensional submanifold of B^3 with $\partial K = K \cap \partial B^3 = 2$ points. Compute the homology of the complement $B^3 - K$ (= an apple minus a wormhole).
6. Recall that two covering spaces $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *isomorphic* if there exists a homeomorphism $\widetilde{\phi} : \widetilde{X} \xrightarrow{\sim} \widetilde{X}'$ such that $p' \circ \widetilde{\phi} = p$. Consider the covering spaces $p : \widetilde{X} \to X$ of the torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?

Geometry/Topology Qualifying Exam

Spring 2009

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. Let $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that the map
$$f : S^2 \to \mathbb{R}^4,\ f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$$
is an immersion and that $f(S^2)$ is diffeomorphic to the projective plane $\mathbb{RP}^2$.

2. Let ω be a closed n-form on $\mathbb{R}^{n+1} - \{0\}$. Prove that ω is exact if and only if $\int_{S^n} \omega = 0$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$.

3. Find all vector fields Z on $\mathbb{R}^2$ which satisfy $[X, Z] = 0$ and $[Y, Z] = 0$, where $X = e^x \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ are vector fields defined on all of $\mathbb{R}^2$.

4. Compute $\pi_n(T^p)$ for all $n \geq 1$, where $T^p = S^1 \times \cdots \times S^1$ (p times) is the p-dimensional torus.

5. Compute $\pi_1(\mathbb{R}^3 - K)$, where $K \subset \mathbb{R}^3$ is the union of the vertical axis $\{x = 0, y = 0\}$ and the unit circle $\{x^2 + y^2 = 1, z = 0\}$.

6. Let X be a compact, oriented surface of genus 2 (without boundary), and let A be a simple closed curve which separates the surface X into two punctured tori, as given in Figure 1 below. Then compute the relative homology groups $H_n(X, A)$ for all $n \geq 0$.

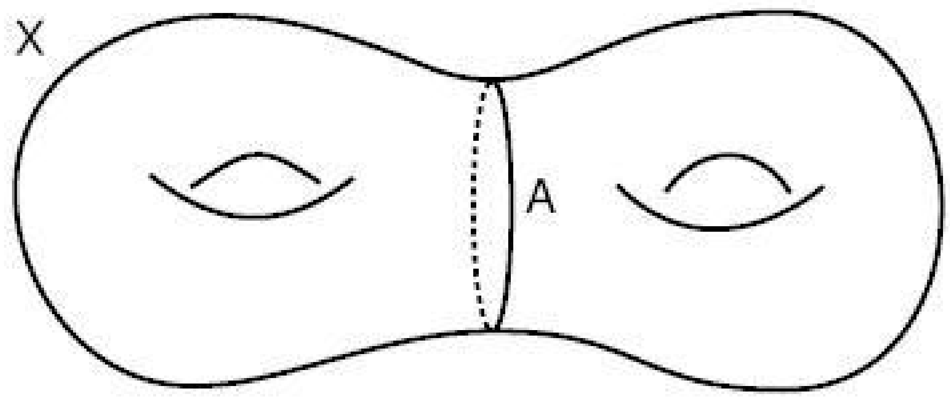


FIGURE 1

Geometry and Topology Graduate Exam
Fall 2009

Problem 1. Let $f : M \to N$ be a map between two compact oriented manifolds of the same dimension. Suppose that the subgroup $f^*(\pi_1(M))$ has finite index in $\pi_1(N)$.

a. Show that the index $[\pi_1(N) : f^*(\pi_1(M))]$ divides the degree of f.
b. Give an example where $[\pi_1(N) : f^*(\pi_1(M))]$ is different from the degree of f.

Problem 2. Is there a differentiable map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends the vector field $\frac{\partial}{\partial x}$ to the vector field $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and sends the vector field $\frac{\partial}{\partial y}$ to the vector field $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$?

Problem 3. Let $f : S^n \to S^n$ be a degree 5 map from the sphere S^n to itself.

a. Show that there exists $x_1 \in S^n$ such that $f(x_1) = -x_1$.
b. Show that there exists $x_2 \in S^n$ such that $f(x_2) = x_2$.

Problem 4. Let M be a compact submanifold of $\mathbb{R}^n$, of dimension at most $n-3$, and let $f : B^2 \to \mathbb{R}^n$ be a differentiable map from the 2–dimensional ball (or disk) B^2 to $\mathbb{R}^n$. Let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation along the vector $v \in \mathbb{R}^n$.

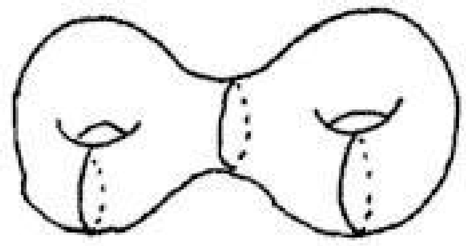
a. Show that there exists arbitrarily small vectors $v \in \mathbb{R}^n$ such that the image of $T_v \circ f$ is disjoint from M.
b. Conclude that the complement $\mathbb{R}^n - M$ is simply connected.

Problem 5. Let ω be a closed form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Show that, for any two differentiable maps $f, g : S^n \to \mathbb{R}^{n+1} - \{0\}$, the ratio

$$\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}$$

is a rational number when the denominator is not 0.

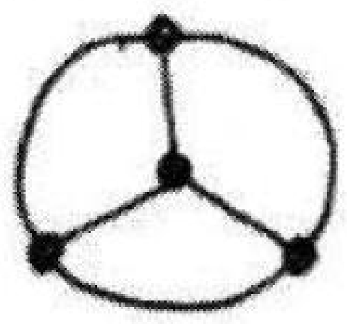
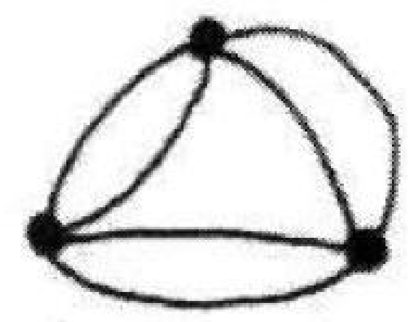
Problem 6. Let S be the standard surface of genus 2 in $\mathbb{R}^3$ as in the picture below, and let W be the closure of the bounded component of $\mathbb{R}^3 - S$. Compute the relative homology groups $H_n(W, S)$.



Problem 7. Let M be a compact connected submanifold of an oriented manifold N, with $\dim M = \dim N - 1$. Show that M is orientable if and only if it admits arbitrarily small connected neighborhoods U such that $U - M$ is disconnected. Namely, if and only if, for every open subset $V \subset N$ containing M, there is a connected open subset $U \subset V$ such that $U - M$ is not connected.

Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$X_1 =$  $X_2 =$ 

Problem 2. Let P_1, P_2, P_3 be three distinct points in the sphere S^2, and let X be the topological space obtained from S^2 by gluing these three points together. Compute all homology groups $H_n(X;\mathbb{Z})$.

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface Σ in $\mathbb{R}^3$ at the point p. Does there exist a compact immersed surface Σ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the *orthogonal group* $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = \mathrm{id}\}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $x_1, x_2, \ldots, x_n$ are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p}\,\omega \ \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: S^{n-1}.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates x_1, $y_1, \ldots, x_n, y_n$. If f is a smooth function on $\mathbb{R}^{2n}$, find the vector field X such that $i_X\omega = df$, where i_X denotes the interior product. Then compute the Lie derivative $\mathcal{L}_X\omega$.

Problem 7. Let X be a topological space such that the homology group $H_p(X;\mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X;\mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}\big(C_{p+1}(X;\mathbb{Z}), \mathbb{Z}\big)$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X, \mathbb{Z})$ with $\partial\alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k}u(\beta)$$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$. Namely, show that $L_u([\alpha])$ is independent of k, β and of the representative α of $[\alpha] \in H_p(X;\mathbb{Z})$.

Geometry/Topology Qualifying Exam

Spring 2011

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Let $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ be the 3-dimensional sphere, oriented as the boundary of the unit ball B^4 in $\mathbb{R}^4$ with the standard orientation. Compute $\int_{S^3} \omega$, where
$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$
(You may leave your answer in terms of volumes $vol(S^n)$ and $vol(B^n)$.)

2. (10 pts) Let $M = \{(x, y) \mid x, y \in \mathbb{R}^3, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = 0\}$, where $\langle x, y\rangle$ is the standard inner product on $\mathbb{R}^3$. Show that M is a smooth compact embedded submanifold of $\mathbb{R}^6$ and explain how M can be identified with the unit tangent bundle of S^2.

3. (20 pts) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.
   (a) (5 pts) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$.
   (b) (5 pts) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.
   (c) (5 pts) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.
   (d) (5 pts) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

4. (10 pts) Given a continuous map $f : X \to Y$ between topological spaces, define
$$C_f = \left((X \times [0, 1]) \coprod Y\right) / \sim,$$
where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Then prove that there is a long exact sequence
$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$
where f_* is the map on homology induced from f and $\widetilde{H}_i$ denotes the ith reduced homology group.

5. (10 pts) Prove that the fundamental group of a connected Lie group G is abelian. (A *Lie group* G is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between fg and gf, where f and g are loops in G.]

6. (10 pts) Let $M \subset \mathbb{R}^3$ be an embedded compact oriented surface (without boundary) of genus $g \geq 1$. Show that the Gaussian curvature κ of M must vanish somewhere on M.