Work three of the following four problems. Show your work.

1. (a) State the Poincaré-Bendixon theorem.
   (b) Show that the system
   \[ \begin{align*}
   \dot{x} &= y \\
   \dot{y} &= -x + y(4 - x^2 - 4y^2).
   \end{align*} \]
   has a periodic orbit.

2. Let \( A \in \mathcal{M}_n(\mathbb{R}) \), \( B \in \mathcal{C}(\mathbb{R}^+; \mathcal{M}_n(\mathbb{R})) \), and \( f \in \mathcal{C}(\mathbb{R}^+; \mathbb{R}^n) \). Suppose that the function \( t \mapsto e^{tA} \) is bounded on \( \mathbb{R}^+ \), and
   \[ \int_0^{+\infty} \|B(s)\| \, ds < +\infty, \quad \int_0^{+\infty} \|f(s)\| \, ds < +\infty. \]
   (a) Prove that all solutions of the system
   \[ x' = (A + B(t))x + f(t) \]
   are bounded.
   (b) Prove that all solutions of the system
   \[ x' = B(t)x \]
   have a finite limit as \( t \to +\infty \).
   (c) Suppose now that the function \( t \mapsto e^{tA} \) is bounded on \( \mathbb{R} \). Prove that if \( x(\cdot) \) is a solution of the system \( x' = (A + B(t))x \), then the function \( y(t) = e^{-tA}x(t) \) has a finite limit when \( t \to +\infty \).
   (d) (Application:) (Extra Credit) Let \( q : \mathbb{R}^+ \to \mathbb{R} \) be a continuous function such that
   \[ \int_0^{+\infty} |q(t)| \, dt < +\infty \]
   and \( x \) a solution of
   \[ x'' + (1 + q(t))x = 0. \]
   Prove the existence of two constants \( \alpha, \beta \in \mathbb{R} \) such that
   \[ \lim_{t \to +\infty} [x(t) - \alpha \cos t - \beta \sin t] = 0. \]

3. Consider the second order ordinary differential equation
   \[ x'' + p(t)x' + ax = 0 \]
   where \( a > 0 \) and \( \int_0^t p(s) \, ds \to \infty \) an \( t \to \infty \). Suppose \( \phi(t) \) and \( \psi(t) \) form a fundamental set of solutions, i.e.
   \[ X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix} \]
   is non-singular. Prove \( \det X(t) \to 0 \) as \( t \to \infty \).
4. Consider the scalar differential equation

\[ \ddot{x} + f(t)x = 0, \]

where \( f(t + T) = f(t) \) is periodic with period \( T \).

(a) Define the monodromy matrix, \( C \), for the \( T \)-periodic system and show that \( \det C = 1 \).

(b) Define the Floquet multipliers, denoted \( \mu_1, \mu_2 \), associated with \( C \) and show they satisfy

\[ \mu_i^2 - \tau \mu_i + 1 = 0, \quad i = 1, 2, \]

where \( \tau := \text{tr} C \).

(c) Show that if \( |\tau| < 2 \), then all solutions \( x(t) \) remain bounded as \( t \to +\infty \).
DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2018

Each problem is worth 20 points. There are 4 problems.

1. Consider the system of ODEs,

\[
\begin{align*}
x' &= \mu x - y - (x^2 + y^2)x, \\
y' &= x + \mu y - (x^2 + y^2)y,
\end{align*}
\]

where \( \mu \in (-1, 1) \).

(a) Show that trajectories never leave the box \([-A, A] \times [-B, B]\) for suitably chosen \(A\) and \(B\).

(b) Show that for \( \mu > 0 \) there exists a non-trivial closed trajectory \( \Gamma \) in the plane.

(c) Describe the bifurcation that takes place as \( \mu \) increases through \( \mu = 0 \).

(d) Compute the approximate period of the closed cycle for \( \mu = 0.001 \).

(e) Show that \( \Gamma \) in part (b) is locally exponentially attracting, i.e. nearby trajectories decay exponentially toward \( \Gamma \).

2. Consider the system

\[
\begin{align*}
x'(t) &= -y(t)z(t) \\
y'(t) &= x(t)z(t) \\
z'(t) &= -x(t)y(t).
\end{align*}
\]

(a) Show that the quantity \( x^2(t) + 4y^2(t) + 9z^2(t) \) is conserved, i.e constant along orbits.

(b) Find all stationary points and determine their type and stability.

3. For the system

\[
\begin{align*}
x' &= 3(\sin^2 t)x - 3y \\
y' &= -(\sin^4 t)x - y,
\end{align*}
\]

(a) Show there is at least one solution \( \psi(t) = (x(t), y(t)) \) that satisfies \( |\psi(t)| \rightarrow \infty \) as \( t \rightarrow \infty \).

(b) Show there can be no solution \( \phi(t) = (x(t), y(t)) \) that satisfies \( x(t_0) = x'(t_0) = 0 \) for some \( t_0 \in \mathbb{R} \) except the identically zero solution.

4. For the system of

\[
\begin{align*}
x' &= 2x + 5y + 5x^2 - 4y^2 \\
y' &= 3x + y - 6x^2 + 5y^2,
\end{align*}
\]

(a) Show there is at least one non-identically zero solution \( \psi(t) = (x(t), y(t)) \) that satisfies \( |\psi(t)| \rightarrow 0 \).

(b) Is the origin asymptotically stable?