DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2017

Work three of the following four problems. Show your work.

1. (a) State the Poincaré-Bendixon theorem.

   (b) Show that the system

$$\dot{x} = y$$
$$\dot{y} = -x + y(4 - x^2 - 4y^2).$$

   has a periodic orbit.

2. Let $A \in \mathcal{M}_n(\mathbb{R}),\ B \in \mathcal{C}(\mathbb{R}^+; M_n(\mathbb{R}))$, and $f \in \mathcal{C}(\mathbb{R}^+; \mathbb{R}^n)$. Suppose that the function $t \mapsto e^{tA}$ is bounded on $\mathbb{R}^+$, and

$$\int_0^{+\infty} ||B(s)||\, ds < +\infty, \qquad \int_0^{+\infty} ||f(s)||\, ds < +\infty.$$

   (a) Prove that all solutions of the system

$$x' = (A + B(t))x + f(t)$$

   are bounded.

   (b) Prove that all solutions of the system

$$x' = B(t))x$$

   have a finite limit as $t \to +\infty$.

   (c) Suppose now that the function $t \mapsto e^{tA}$ is bounded on $\mathbb{R}$. Prove that if $x(\cdot)$ is a solution of the system $x' = (A + B(t))x$, then the function $y(t) = e^{-tA}x(t)$ has a finite limit when $t \to +\infty$.

   (d) (Application:) (Extra Credit) Let $q : \mathbb{R}^+ \mapsto \mathbb{R}$ be a continuous function such that

$$\int_0^{+\infty} |q(t)|\, dt < +\infty$$

   and x a solution of

$$x'' + (1 + q(t))x = 0.$$

   Prove the existence of two constants $\alpha,\ \beta \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} [x(t) - \alpha \cos t - \beta \sin t] = 0.$$

3. Consider the second order ordinary differential equation

$$x'' + p(t)x' + a\, x = 0$$

   where $a > 0$ and $\displaystyle\int_0^t p(s)ds \to \infty$ an $t \to \infty$. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e.

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

   is non-singular. Prove $\det X(t) \to 0$ as $t \to \infty$.

4. Consider the scalar differential equation

$$\ddot{x} + f(t)x = 0,$$

where $f(t+T) = f(t)$ is periodic with period T.

(a) Define the monodromy matrix, C, for the T-periodic system and show that $\det C = 1$.

(b) Define the Floquet multipliers, denoted μ_1, μ_2, associated with C and show they satisfy

$$\mu_i^2 - \tau\mu_i + 1 = 0, \quad i = 1, 2,$$

where $\tau := \operatorname{tr} C$.

(c) Show that if $|\tau| < 2$, then all solutions $x(t)$ remain bounded as $t \to +\infty$.

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Each problem is worth 20 points. There are 4 problems.

1. Consider the system of ODEs,

$$\begin{aligned} x' &= \mu x - y - (x^2 + y^2)x, \\ y' &= x + \mu y - (x^2 + y^2)y, \end{aligned} \tag{1}$$

   where $\mu \in (-1, 1)$.

   (a) Show that trajectories never leave the box $[-A, A] \times [-B, B]$ for suitably chosen A and B.

   (b) Show that for $\mu > 0$ there exists a non-trivial closed trajectory Γ in the plane.

   (c) Describe the bifurcation that takes place as μ increases through $\mu = 0$.

   (d) Compute the approximate period of the closed cycle for $\mu = 0.001$.

   (e) Show that Γ in part (b) is locally *exponentially* attracting, i.e. nearby trajectories decay exponentially toward Γ.

2. Consider the system

$$\begin{aligned} x'(t) &= -y(t)z(t) \\ y'(t) &= x(t)z(t) \\ 3z'(t) &= -x(t)y(t). \end{aligned}$$

   (a) Show that the quantity $x^2(t)+4y^2(t)+9z^2(t)$ is conserved, i.e constant along orbits.

   (b) Find all stationary points and determine their type and stability.

3. For the system

$$\begin{aligned} x' &= 3(\sin^2 t)x - 3y \\ y' &= -(\sin^4 t)x - y, \end{aligned}$$

   (a) Show there is at least one solution $\psi(t) = (x(t), y(t))$ that satisfies $|\psi(t)| \to \infty$ as $t \to \infty$,

   (b) Show there can be no solution $\phi(t) = (x(t), y(t))$ that satisfies $x(t_0) = x'(t_0) = 0$ for some $t_0 \in \mathbb{R}$ except the identically zero solution.

4. For the system of

$$\begin{aligned} x' &= 2x + 5y + 5x^2 - 4y^2 \\ y' &= 3x + y - 6x^2 + 5y^2, \end{aligned}$$

   (a) Show there is at least one non-identically zero solution $\psi(t) = (x(t), y(t))$ that satisfies $|\psi(t)| \to 0$.

   (b) Is the origin asymptotically stable?