Topics for the Graduate Exam in Differential Equations

Most of the following topics are normally covered in the courses Math 555a and 565a.

This is a two hour exam.

Existence, uniqueness and dependence of initial data and continuation of solutions.

Linear systems, periodic linear systems, Floquet's Theorem, stability of critical points and periodic orbits. Dichotomies, perturbations of linear systems, Lyapunov functions.

2-dimensional systems, classification of elementary critical points, Poincare-Bendixson Theorem, flows, invariant sets. Anosov diffeomorphisms, Stable Manifold Theorem, invariant manifolds, bifurcations and skew-product dynamical systems.

First order equations (the Cauchy problem, method of characteristics)
Sobolev spaces (imbedding theorem, Rellich compactness theorem)
Laplace equation (mean value property, Harnack principle, maximum principle, Liouville's theorem)
Heat equation (Cauchy problem, energy equality, maximum principle, nonhomogeneous heat equation)
Wave equation (D'Alamert's formula, energy equality, Duhamel's principle)

References:

P. Hsieh and Y. Sibuya, Basic Theory of ODE, Springer
J. Hale, ODE, Wiley
E.A. Coddington and N. Levinson, Theory of ODE

F. John, Partial Differential Equations, 4th Edition, Springer
L.C. Evans, Partial Differential Equations

ODE/PDE Qualifying Exam, May 1997

1) Consider the equation

$$x'' + (a + b\cos t)x = 0$$

and let u, v be solutions such that

$$u(0) = 1,\ u'(0) = 0;\ \ v(0) = 0,\ v'(0) = 1.$$

Set $F(a, b) = u(2\pi) + v'(2\pi)$ and show that if $|F(a, b)| > 2$ then no solutions remain bounded for all (real) t.

2) Consider the system

$$\begin{aligned} \dot{x} &= -x + y \\ \dot{y} &= kx - y - xz \\ \dot{z} &= -z + xy \end{aligned}$$

where $k > 0$. Show that that all solutions exist and remain bounded for $t \geq 0$.

3) Show that $x' + x = x^2$ has a solution $u(t)$ with $u(0) = x_0$ that satisfies for all $t \geq 0$

$$|u(t)| < 2|x_0|$$

provided $|x_0|$ is small enough.

4) Consider $\dot{x} = F(t, x), x(t_0) = x_0$, where F is continuous and satisfies a Lipschitz condition in some open neighborhood of $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Show that the solution $x = \varphi(t, t_0, x_0)$ satisfies a Lipschitz condition in x_0 near (t_0, x_0).

5) Show that a C^1 solution of the following equation can not exist in a large time interval

$$\begin{cases} u_t + uu_x = 0, & t \geq 0, x \in (-\infty, +\infty), \\ u(x, 0) = h(x), & x \in (-\infty, +\infty), \end{cases}$$

where h is a smooth function with compact support.

6) Show that the solution $U \in C^2(\overline{\Omega})$ of

$$\triangle U = 0 \text{ in } \Omega,\ U = f \text{ on } \partial\Omega$$

minimizes the integral

$$\int_\Omega |\bigtriangledown U|^2 dx$$

among all functions in $C^1(\overline{\Omega})$ with boundary values f.

7a) Let $f(x)$ be bounded and continuous for $x \in R^n$ and satisfy

$$\int |f(y)| dy < \infty.$$

Show that there exists a solution $u(x,t)$ of

$$\begin{cases} u_t - \Delta u = 0 \text{ for } x \in R^n, t > 0, \\ u(x,0) = f(x), \end{cases}$$

such that $\lim_{t\to\infty} u(x,t) = 0$.

7b) Let $n = 1$. Show that the same conclusion holds for $f \in C^2(R)$ that have period 2π and satisfy

$$\int_0^{2\pi} f(y)dy = 0.$$

8) Let $u = u(x_1, x_2)$ be a solution of class C^2 on the semi-strip

$$x_1 \geq 0, \;\; a \leq x_2 \leq b,$$

of the equation

$$u_{x_1} - u_{x_2 x_2} = 0.$$

Show that u is determined uniquely by its Cauchy data on

$$x_1 = 0, a < x_2 < b.$$

ODE/PDE Qualifying Exam
May 4, 1998

1. Consider $\dot{x} = [A + \varepsilon B(t)]x$ where A is an $n \times n$ real matrix whose eigenvalues λ satisfy $Re\lambda < 0$ and $B(t)$ is continuous and periodic with period 1.

Show that for $|\varepsilon|$ sufficiently small, all solutions satisfy $\lim_{t\to\infty} |x(t)| = 0$.

2. Show that solutions of

$$\dot{x} = \frac{t^2 x^5}{1 + x^2 + x^4}$$

can be continued to the whole real line.

3. Let f and g be C^1 vector fields in $\mathbb{R}^2$ such that the inner product $\langle f(x), g(x)\rangle = 0$ for all x. If f has a closed orbit prove g has a zero.

4. Solve

$$(y+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y$$
$$u(x,1) = 1 + x.$$

Show that this solutions is unique in the class $C^1(\Omega)$ where $\Omega = \{(x,y) \in \mathbb{R}^2, y > 0\}$.

5. Let u be a harmonic function in a bounded connected open set $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. Show that there exists $x_0 \in \partial\Omega$ such that

$$|\nabla u(x_0)|^2 = \max_{x\in\overline{\Omega}} |\nabla u(x)|^2.$$

6. Let $u \in C^2(\mathbb{R}^n \times \mathbb{R})$ be a solution of

$$\frac{\partial^2 u}{\partial t^2} = \triangle u$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

where $f, g \in C^\infty(\mathbb{R}^n)$ have compact support. Compute $\int_{\mathbb{R}^n} u(x,t)dx$ for every $t \in \mathbb{R}$.

1. Assume that $u \in C^2(\mathbb{R}^3)$ is a harmonic function such that

$$|u(x)| \leq C(|x|^{1/2} + 1), \quad x \in \mathbb{R}^3$$

for some $C > 0$. Prove that u is a polynomial.

2. Assume that $u(x,t)$ is such that

$$u \in C^\infty(\mathbb{R} \times (0,\infty)) \cap C(\mathbb{R} \times [0,\infty)) \cap L^\infty(\mathbb{R} \times (0,\infty))$$

and

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = u_0(x) \end{aligned}$$

where

$$u_0(x+1) = u_0(x), \quad x \in \mathbb{R}.$$

Prove that

$$u(x+1,t) = u(x,t) \text{ for } (x,t) \in \mathbb{R} \times [0,\infty).$$

3. Let $u \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ be a solution of the equation

$$\begin{aligned} &u_{tt} = \Delta u \\ &u(x,0) = 0, && x \in \mathbb{R}^3 \\ &u_t(x,0) = g(x), && x \in \mathbb{R}^3. \end{aligned}$$

If the support of g is compact, prove that

$$\lim_{t \to +\infty} ||u(\cdot,t)||_{L^\infty} = 0.$$

4. Consider the equation

$$\varphi'' + f\varphi' + g\varphi = 0$$

where $f(z), g(z)$ are complex functions. What conditions must f and g satisfy if ∞ is a regular point.

Assume f, g are not constant, and ∞ regular. Show that 0 is a singular point.

5. Consider the following system of differential equations

$$\begin{aligned} x' &= y + \alpha \sin x \\ y' &= -\beta x. \end{aligned}$$

Linearize at the critical points and classify the critical points.

6. Let $A(t)$ be a real continuous matrix for $t \in \mathbb{R}^+$ such that

$$\lim_{t \to \infty} A(t) = C$$

where C is a constant matrix. Show that every solution $\vec{x}(t)$ of $\vec{x}' = A\vec{x}$ has the property that

$$\lim_{t \to \infty} \frac{1}{t} \log ||\vec{x}(t)|| = \mu$$

where μ is the real part of an eigenvalue of C.

Ordinary and Partial Differential Equations - Spring '99

Show your work and explain to obtain full credit

1. Show that for all ϵ every solution of

$$\begin{aligned} x' &= y \\ y' &= -x - \epsilon y \end{aligned}$$

goes to zero as $t \to \infty$.

2. Consider the second order differential equation $x'' + (a + b\sin t)x = 0$ and let u, v be two solutions with the initial data $u(0) = 1, u'(0) = 0, v(0) = 0, v'(0) = 1$. Show that if $u(2\pi) + v'(2\pi) = 2$ then there exists at least one periodic solution with period 2π.

3. Show that the trivial solution of

$$u'' + (u^2 - 2u + 1)u' + u^3 - u^5 = 0$$

is asymptotically stable.

4. a) Find the solution $u(x,t)$, $t \geq 0$, $x \geq 0$ of

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & t \geq 0,\ x \geq 0, \\ u(0,t) = h(t), & t \geq 0, \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & x \geq 0, \end{cases}$$

where $f,\ g,\ h \in C^2([0,\infty))$.
b) Let $f = g = 0$ and $h(t) = \sin \pi t$. Is the solution a continuous and differentiable function? For each t, determine the set $\{x : u(x,t) \neq 0\}$.

5. (Dirichlet principle). Show that a solution $v \in C^2(\bar{D})$ of

$$\Delta v = 0 \text{ in } \bar{D}, \quad v = f \text{ on } \partial D$$

minimizes the Dirichlet integral $\int_D |\nabla u|^2\, dx$ among all functions $u \in C^1(\bar{D})$ with boundary value f.

6. A. Consider the following problem for $u(x,y)$, $y \geq 0$, $x \in (-\infty, \infty)$,

$$\begin{cases} u_y = u_{xx}, \\ u(x,0) = f(x), \end{cases}$$

where $f \in C^\infty$.
a) Is the line $y = 0$ non characteristic?
b) Are the derivatives of u determined uniquely on $y = 0$ provided they exist?
B. Given a 2π-periodic function $f \in C^2(\mathbf{R})$, solve the Cauchy problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & t > 0,\ x \in (-\infty, \infty), \\ u(x,0) = f(x), & x \in (-\infty, \infty); \end{cases}$$

Find $\lim_{t\to\infty} u(x,t)$.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM , January 2004

1. For the system

$$\frac{dx}{dt} = 3x - xy - x^2 = f(x,y)$$
$$\frac{dy}{dt} = -y + xy - y^2 = g(x,y),$$

   (a) Find all stationary points

   (b) Discuss the stability behavior in a small neighborhood of the stationary points (existence of the stationary points, their stability, stable and unstable manifolds).

   (c) By applying general theorems describe the modified behavior (existence of the stationary points, their stability, stable and unstable manifolds) near the stationary points if we consider instead,

$$\frac{dx}{dt} = f(x,y) + \epsilon\phi(x,y)$$
$$\frac{dy}{dt} = g(x,y) + \epsilon\psi(x,y),$$

   where $\phi, \psi \in C^1$ and $\epsilon > 0$ is sufficiently small.

   (d) Can you think of a way of defining ϕ and ψ in order to create a heteroclinic orbit?

2. Contrast the bifurcations which take place in the examples

$$\frac{dx}{dt} = \delta x - x^3 \qquad \text{(Supercritical case)}$$
$$\frac{dy}{dt} = \delta x + x^3 - x^5 \qquad \text{(Subcritical case)},$$

   as the parameter δ passes through the bifurcation values.

3. (a) Show that the ω-limit set of a bounded orbit of

$$\frac{dx}{dt} = f(x), \qquad f \in C^1, \quad x \in \mathbb{R}^n,$$

   is a compact and connected set.

   (b) Show by example that under the same assumptions, one of the above properties (compactness or connectedness) is lost when we consider instead a difference equation

$$x_{n+1} = f(x_n), \qquad f \in C^1, \quad x \in \mathbb{R}^n$$

4. Let u be a smooth solution of

$$\Delta u = f, \quad \text{in } B(0,1), \quad \text{and} \quad u = g, \quad \text{on } \partial B(0,1),$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^n$.

Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|)$$

5. Assume that H is a smooth function on $\mathbb{R}^{n+1}$, and consider the partial differential equation

$$F = \frac{\partial u}{\partial t} + H(x_1, \ldots, x_n, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) = 0$$

for $u = u(x_1, \ldots, x_n, t)$ and

$$u(x_1, \ldots, x_n, 0) = u_0(x_1, \ldots, x_n).$$

where u_0 is a smooth given function.

Prove the existence of $T > 0$, such that the solution $u = u(x_1, \ldots, x_n, t)$ exists for $0 \leq t < T$.

6. Let $u = u(t,x)$, $0 \leq t \leq T$, $x \in \mathbb{R}$, be a classical solution of the equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(a(t,x)\frac{\partial u(t,x)}{\partial x}\right) + b(t,x)\frac{\partial u(t,x)}{\partial x} + c(t,x)u(t,x) + \frac{\partial f(t,x)}{\partial x}$$

with a given initial condition $u(0,x) = u_0(x)$ and a known function f. Assume that f is infinitely differentiable and compactly supported in $(0,T) \times \mathbb{R}$, u_0 is infinitely differentiable and compactly supported in $\mathbb{R}$, and there exist positive numbers C and δ so that, for all $(t,x) \in (0,T) \times \mathbb{R}$, $a(t,x) \geq \delta$ and $|a(t,x)| + |b(t,x)| + |c(t,x)| \leq C$.

Show that there exists a positive number C_0, depending only on δ, C, and T, so that,

$$\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |u(t,x)|^2 dx + \int_0^T \int_{-\infty}^{+\infty} \left|\frac{\partial u(t,x)}{\partial x}\right|^2 dx dt$$

$$\leq C_0 \left(\int_{-\infty}^{+\infty} |u_0(x)|^2 dx + \int_0^T \int_{-\infty}^{+\infty} |f(t,x)|^2 dx dt\right).$$

Note. You are welcome to assume any smoothness of the functions a, b, c and all the necessary integrability of u and its derivatives.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM , September 2004

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded smooth function. True or false: every solution of the equation

$$\frac{dx(t)}{dt} = f(x(t))$$

is either strictly increasing or strictly decreasing, as a function of t. Give a proof or a counterexample.

Hint: when does the derivative of x vanish?

2. Describe the bifurcation which take place as μ varies on the interval $(-1, 1)$:

$$\frac{dz}{dt} = z(\mu + i) - (2 - i)(z^2\overline{z}), \qquad z = x + iy$$

and plot the bifurcation diagram.

3. Consider the following system of ODE's

$$\begin{aligned} x' &= x + 2y - x^2 - y^2 + x^3y - x\sin xy \\ y' &= 4x + 3y + x^2y^2 - xy\exp(x + y) \end{aligned}$$

Show that in every neighborhood of $(x, y) = (0, 0)$ there are solutions $(x(t), y(t))$ that remain in the neighborhood and

$$\lim_{t\to\infty}(x(t), y(t)) = (0, 0).$$

For a typical neighborhood of $(0, 0)$ draw an approximation to the set of all such solutions. Is it possible to find a small enough neighborhood of $(0, 0)$ so that ALL solutions in the neighborhood remain in the neighborhood and satisfy the above "limit" condition? Why or why not?

4. Suppose that $u = u(t, x)$ is a smooth solution of the heat equation

$$u_t - \Delta u + c(t, x)u = 0, \ t > 0, \ x \in D; \ u(0, x) = g(x),$$

in a smooth bounded domain $D \subset \mathbb{R}^n$ with zero boundary conditions. Assume that g is a non-negative smooth function and c is a bounded smooth function. Show that $u(t, x) \geq 0$ for all $t \geq 0$ and all $x \in D$.

Suggestion: for a suitable number a, consider $v(t, x) = e^{at}u(t, x)$ and recall the maximum principle.

5. Let u be a smooth solution of

$$\begin{aligned} u_{tt} - \Delta u = 0 \quad &\text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \ u_t = h \quad &\text{on } \mathbb{R}^3 \times \{t = 0\}, \end{aligned}$$

where g and h are smooth and have compact support. Prove the existence of a positive constant C such that

$$|u(x, t)| \leq \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

6. Let U be a smooth bounded domain of $\mathbb{R}^n$. Prove the existence of a constant C, such that for all $u \in W_0^{2,p}(U), p \geq 1$, we have

$$\int_U |Du|^p \, dx \leq C\left(\int_U |u|^p \, dx\right)^{1/2} \left(\int_U |D^2u|^p \, dx\right)^{1/2}.$$

(Hint: First prove the inequality for smooth compactly supported functions).

DIFFERENTIAL EQUATIONS QUALIFYING EXAM, February 2005

1. Consider a smooth function $f = f(t, x) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $t \geq 0$. Assume that there exist real numbers a_1, a_2 so that

$$x \cdot f(t, x) \leq a_1 |x|^2 + a_2$$

for all $t \geq 0$, $x \in \mathbb{R}^n$.

Show that, for every $x_0 \in \mathbb{R}^n$ and every $T > 0$, there exists a unique solution $x = x(t)$ of the equation

$$\frac{dx(t)}{dt} = f(x(t), t), 0 < t \leq T, \ x(0) = x_0.$$

(As usual, $|x|$ denotes the Euclidean norm of x, and $x \cdot f$ is the inner product in $\mathbb{R}^n$)

Suggestion: use Gronwal's lemma to get a bound on $|x(t)|^2$.

2. Describe the bifurcation which take place as μ varies on the interval $(-1, 1)$:

$$\frac{dz}{dt} = z(\mu + i) - (2 - i)(z^2 \overline{z}), \qquad z = x + iy$$

and plot the bifurcation diagram.

3. Consider the following system of ODE's

$$\begin{aligned} x' &= x + 2y - x^2 - y^2 + x^3 y - x \sin xy \\ y' &= 4x + 3y + x^2 y^2 - xy \exp(x + y) \end{aligned}$$

Show that in every neighborhood of $(x, y) = (0, 0)$ there are solutions $(x(t), y(t))$ that remain in the neighborhood and

$$\lim_{t \to \infty} (x(t), y(t)) = (0, 0).$$

For a typical neighborhood of $(0, 0)$ draw an approximation to the set of all such solutions. Is it possible to find a small enough neighborhood of $(0, 0)$ so that ALL solutions in the neighborhood remain in the neighborhood and satisfy the above "limit" condition? Why or why not?

4. Suppose that $u = u(t, x)$ is a smooth solution of the heat equation

$$u_t - \Delta u + c(t, x) u = 0, \ t > 0, \ x \in D; \ u(0, x) = g(x),$$

in a smooth bounded domain $D \subset \mathbb{R}^n$ with zero boundary conditions. Assume that g is a non-negative smooth function and c is a bounded smooth function. Show that $u(t, x) \geq 0$ for all $t \geq 0$ and all $x \in D$.

Suggestion: for a suitable number a, consider $v(t, x) = e^{at} u(t, x)$ and recall the maximum principle.

5. Let u be a smooth solution of

$$\begin{aligned} &u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\ &u = g, \ u_t = h \quad \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{aligned}$$

where g and h are smooth and have compact support. Prove the existence of a positive constant C such that

$$|u(x, t)| \leq \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

6. Let U be a smooth bounded domain of $\mathbb{R}^n$. Prove the existence of a constant C, such that for all $u \in W_0^{2,p}(U)$, $p \geq 2$, we have

$$\int_U |Du|^p \, dx \leq C \left(\int_U |u|^p \, dx \right)^{1/2} \left(\int_U |D^2 u|^p \, dx \right)^{1/2}.$$

(Hint: First prove the inequality for smooth compactly supported functions).

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2006

1. Let $f : R^n \to R^n$ be Lipshitz continuous and $x(t)$ a solution to the initial value problem $x' = f(x)$, $x(t_0) = x_0 \neq 0$. Show that if $x \cdot f(x) \geq \|x\|^3$ then the solution cannot extend to the interval $[t_0, \infty)$.

2. Consider the system $x' = h(t)Ax$, where $x(t)$ is an n-vector, A is a constant $n \times n$ matrix and $h(t)$ is strictly positive and continuous. Show that the trivial solution 0 is asymptotically stable if all eigenvalues of A lie in the left half plane and the integral $\int_0^\infty h(t)dt$ diverges. Also give an example that shows that the stability may not apply if the integral converges.

3. Show that at $(0,0)$ in the x, y plane, NO analytic Center Manifold, $y = f(x)$ exists for the following system:

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y + x^2. \end{aligned}$$

4. Let u be harmonic and bounded on $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n);\ x_n > 0\}$ and $u = 0$ on $\{(x_1, \ldots, x_n);\ x_n = 0\}$. Show that u is a constant.

5. Let $g(x)$ be a bounded and continuous function on $\mathbb{R}^n$, and

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y)\, dy.$$

   (i) What is the PDE satisfied by $u(x,t)$?

   (ii) Show that $|u(x,t)| \leq \sup_y |g(y)|$.

   (iii) If in addition, $\int_{\mathbb{R}^n} |g(y)|\, dy < \infty$, show that $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in \mathbb{R}^n$.

6. Let u be a smooth solution of

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty) \qquad (1)$$

$$u = g,\ u_t = h \quad \text{on } \mathbb{R}^3 \times \{t = 0\} \qquad (2)$$

   where g and h are smooth and have compact support. Prove the existence of a positive constant C such that

$$|u(x,t)| \leq \frac{C}{t}$$

   for all $x \in \mathbb{R}^3$ and $t > 0$.

   Hint:

$$u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} th(y) + g(y) + \nabla g(y) \cdot (y - x)\, dS(y)$$

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2008

1. a) Solve the linear partial differential equation

$$e^x u_x + u_y = u \quad \text{with } u(x,0) = g(x).$$

b) Solve the nonlinear partial differential equation

$$x^2 u_x + y^2 u_y = u^2 \quad \text{with } u = 1 \text{ on the line } y = 2x$$

2. Let $c \in \mathbb{R}$. Write down an explicit formula for a solution of

$$\begin{aligned} u_t - \Delta u + cu &= f \qquad && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g && \text{on } \mathbb{R}^n \times t = 0. \end{aligned}$$

3. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$.

a) If

$$u(x) = |x|^{-\alpha}, \quad x \in B(0,1)$$

For what values of α, n, p the function u is in the Sobolev space $W^{1,p}(B(0,1))$.

b) If $u(x) = \ln\ln(1 + \frac{1}{|x|})$, for $x \in B(0,1)$, Prove that $u \in W^{1,n}(B(0,1)))$ but not in $L^\infty(B(0,1))$.

4. Let $g : R^2 \times (-1, 1) \to R^2$ be of class C^4 and consider the mapping $x \to g(x, \mu)$ where $g(0, \mu) \equiv 0$ and $\frac{\partial g}{\partial x}(0, \mu)$ has complex eigenvalues $\lambda(\mu), \bar{\lambda}(\mu)$ that leave the unit circle as μ increases through 0, i.e. $|\lambda(0)| = 1$ and $\frac{d|\lambda(\mu)|}{d\mu} > 0$. After a linear transformation and letting $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, the mapping takes the form

$$z \to \lambda(\mu)z + \dots .$$

   (a) State conditions on $\lambda(0)$ that allow the mapping to be transformed into the *Normal Form*

$$w \to we^{\alpha(\mu)+\beta(\mu)|w|^2} + \mathcal{O}(|w|^4). \tag{1}$$

   (b) Give a condition on $\beta(0)$ that guarantees a Neimark-Sacker bifurcation of the origin into an asymptotically stable invariant curve $\Gamma(\mu)$ for $0 < \mu < \mu*$ surrounding the origin.

   (c) Write down an expression for the first approximation to Γ. **Hint:** Drop the $\mathcal{O}(|w|^4)$ terms and separate real and imaginary parts in the expression in the exponent of (1).

5. Consider the vector field

$$\begin{aligned} x' &= x^2 y - x^5 \\ y' &= -y + x^2. \end{aligned}$$

Near $(0, 0)$ there is a Center Manifold, the graph of $y = h(x) = ax^2 + bx^3 + \dots$. Write down the first order differential equation satisfied by the function h and find a and b.

6. Consider the product space $\mathbb{R}^n \times \mathbb{R}$ and let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x, t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^n$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

and $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = f(t, y), \quad y(t_1) = x$, evaluated at t_2. Assume that $f \in C^1$ and the divergence,

$$\text{div } f \doteq \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2} + \dots + \frac{\partial f_n}{\partial y_n} = 0.$$

Show (1) T is a diffeomorphism and (2) T is (Lebesgue)measure preserving.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2008

1. Consider the linear p-periodic system in $\mathbb{R}^n$ with A continuous,

$$x'(t) = A(t)x(t), \qquad A(t+p) = A(t).$$

   (a) State Floquet's Theorem,

   (b) Assume each eigenvalue, $\lambda(t)$ of $A(t)$ satisfies

$$\mathcal{R}e\lambda(t) \leq -1.$$

   What can be concluded about the asymptotic stability of the stationary solution, $x(t) \equiv 0$?

   (c) Describe with a sketch the skew-product dynamical system defined by the above system.

2. Show that if real valued continuos functions $f(x)$, $g(x)$, and $h(x)$ satisfy the inequalities

$$f(x) \geq 0, \quad g(x) \leq h(x) + \int_0^x f(\xi)g(\xi)\,d\xi \tag{1}$$

on an interval $0 \leq x \leq x_0$, then

$$g(x) \leq h(x) + \int_0^x \left\{ f(\zeta)h(\xi) \exp\left[\int_\xi^x f(\eta)\,d\eta \right] \right\} d\xi$$

on $0 \leq x \leq x_0$.

Hint: If we put

$$y(x) = \int_0^x f(\xi)g(\xi)\,d\xi,$$

then $\dfrac{dy}{dx} \leq f(x)h(x) + f(x)y$ and $y(0) = 0$.

3. Let $\alpha \in [-1, 1]$, $\beta > 0$ and $z = x + iy$ a complex variable. Consider the system in $\mathbb{R}^2$,

$$z' = (\alpha + i\beta)z - |z|^4 z.$$

Describe in detail the change that takes place in the phase portrait as α varies from -1 to 1. Prove your statements.

4. Let $u(x,t)$ be a bounded solution to the Cauchy Problem for the Heat equation

$$u_t = a^2 u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \quad a > 0$$
$$u(x,0) = \varphi(x).$$

Here $\varphi(x) \in C(\mathbb{R})$ satisfies

$$\lim_{x\to\infty} \varphi(x) = b, \quad \lim_{x\to-\infty} \varphi(x) = c$$

Compute the limit of $u(x,t)$ as t goes to infinity. Justify your argument carefully.

Hint: Use the explicit form of the solution, a change of variables $z = (y-x)/\sqrt{4a^2t}$ and a splitting of the integral into three parts.

5. Let $\varphi(x)$ be a function in $C_0^\infty(\mathbb{R}^3)$, and consider the following damped wave equation

$$u_{tt} - \Delta u + \varphi(x) u_t = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}$$
$$u|_{t=0} = f, \quad u_t|_{t=0} = g.$$

a) Fix $x_0 \in \mathbb{R}^3$ and $t_0 > 0$. Let $C = \{(x,t) : 0 \le t \le t_0,\ |x - x_0| \le |t - t_0|\}$ be the cone of dependence for the point (x_0, t_0) and define $B_\tau = C \cap \{t = \tau\}$. Prove that the energy

$$e(\tau) = \frac{1}{2} \int_{B_\tau} (|u_t|^2 + |\nabla u|^2)\, dt$$

is decreasing and that if $u = u_t = 0$ in B_0 then $u = 0$ identically on C.

b) Use the fact that $f,\ g \in C_0^\infty(\mathbb{R}^3)$ to conclude that the energy of the solution

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|u_t|^2 + |\nabla u|^2)\, dt$$

is decreasing.

6. Find the general solution of the equation

$$x u_{xx} + u_{xy} = 0; \quad u = u(x; y)$$

7. (a) Let $U \subset \mathbb{R}^n$ be an open bounded domain. Let u; v be two harmonic functions in U, which are continuous in U. Show that if $u \le v$ on the boundary ∂U of U, then $u \le v$ in U.

(b) Consider the domain $D = \{x \in \mathbb{R}^n : |x| < 1\} \backslash \{0\}$, where $n \ge 2$. Suppose that u is a harmonic function in D, it is continuous in D, and $u = 0$ on the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. Prove that $u \equiv 0$ in D.

Hint: Compare u with functions of the form $const \cdot \Phi(x)$ on domains of the form $\{x \in \mathbb{R}^n : r < |x| < 1\}$, where $\Phi(x)$ denotes the fundamental solution on $\mathbb{R}^n$.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2009

1. Consider the initial value problem for $x, y \in \mathbb{R}$,

$$\frac{dy}{dx} = 2(2+x)(1+y^2), \qquad y(0) = 0.$$

Determine the value of x, if one exists, at which the solution attains its minimum value.
Hint: Solve the equation and study the maximal interval of existence

2. Consider the systems, for $t \in \mathbb{R}^+$,

$$\frac{dy}{dt} = A(t)y, \qquad \text{and} \tag{1}$$

$$\frac{dy}{dt} = [A(t) + B(t)]y. \tag{2}$$

Assume

$$\int_0^\infty \|B(t)\| dt < \infty, \qquad \text{and} \qquad \liminf_{t\to\infty} \int_0^t trA(s)ds > -\infty.$$

If all the solutions of (1) are bounded on $\mathbb{R}^+$ show all the solutions of (2) are bounded on $\mathbb{R}^+$.

3. Let $I = (-\infty, \infty)$ and $\Phi(t) = \begin{pmatrix} t^2 & 1 & 0 \\ t+1 & t^2+1 & 2 \\ t+4 & 0 & t+2 \end{pmatrix}$. Prove that Φ cannot be a fundamental matrix for a linear homogeneous system $x' = A(t)x$ defined on all of I. What happens to the above conclusion when you are allowed to choose the interval $I = (a, b)$, $a < b$?

4. Prove that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p \, dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

5. Solve the Cauchy problem

$$u_x u_y = u, \qquad \text{with } u(0,y) = y^2.$$

6. Let $u(x,t)$ be a solution of the one-dimensional heat equation $u_t = u_{xx}$ with initial data

$$u(x,0) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0. \end{cases}$$

Find

$$a = \lim_{t\to\infty} u(x,t)$$
$$b = \lim_{x\to\infty} u(x,t)$$
$$c = \lim_{x\to-\infty} u(x,t)$$

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2010

The exam has six problems.

1. Consider the product space $\mathbb{R}^n \times \mathbb{R}$ and let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x,t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^n$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

where $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = f(t, y), \quad y(t_1) = x$, evaluated at t_2. Assume that $f \in C^1$ and the divergence,

$$\text{div } f \doteq \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2} + \cdots + \frac{\partial f_n}{\partial y_n} = 0.$$

Show (1) T is a diffeomorphism and (2) T is (Borel) measure preserving. **Note:** Let $(X, \mathcal{B}, \mu)$ be a measurable space. Then $h : X \to X$ is measure preserving if $\mu(h^{-1}(U)) = \mu(U)$ for each $U \in \mathcal{B}$. If h is invertible then h is measure preserving if h^{-1} is measure preserving, i.e. if $\mu(h(U)) = \mu(U)$ for each $U \in \mathcal{B}$.

2. Let

$$\frac{dx}{dt} = f(x), \qquad f \in C^1(\mathbb{R}^3, \mathbb{R}^3)$$

and suppose $x = \phi(t)$ is the solution that satisfies $\|\phi(0)\| = 2$ and $\|\phi(t)\| \le 1$ for $t \ge t_0 > 0$.

   (a) Prove there is a subset of the unit ball that is invariant under the flow generated by the ODE.

   (b) Prove there is a solution $\psi(t)$ that satisfies $\|\psi(0) - \psi(t_n)\| \to 0$ for some sequence $t_n \to \infty$.

3. Consider the planar system

$$\begin{aligned} r' &= r(1-r) \\ \theta' &= r\sin^2\left(\frac{\theta}{2}\right). \end{aligned}$$

   (a) Find the α and ω limit sets of all points in the plane.

   (b) Is the stationary point $(r, \theta) = (1, 0)$ asymptotically stable?

4. Consider the wave equation in $\mathbb{R}^3$

$$u_{tt} - \Delta u = 0 \quad \text{for } x \in \mathbb{R}^3,\ t > 0, \tag{1}$$
$$u(x,0) = 0, \tag{2}$$
$$u_t(x,0) = g(x), \tag{3}$$

where $g \in C_0^\infty(\mathbb{R}^3)$. Prove that there exists a constant C depending only on the given data such that

$$\sup_{x\in\mathbb{R}^3} |u(x,t)| \le \frac{C}{t}, \quad t > 0.$$

5. Use the method of characteristics to solve the following partial differential equation:

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 3u, \quad u(x,0) = u_0(x). \tag{4}$$

6. Let $u(x)$ be harmonic for $x \in \mathbb{R}^3$. Suppose that $\nabla u \in L^2(\mathbb{R}^3)$. Prove that u is a constant.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2011

1. Brouwer's Fixed Point Theorem tells us that if

$$D = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$$

and $f : D \to D$ is continuous then f has a fixed point in D. Obtain this same conclusion for $f \in C^1$ by finding an equilibrium for the vector field $g(x) = f(x) - x$.

2. Consider the autonomous system $x'(t) = f(x(t))$, where $x = (x_1, \ldots, x_n)$ and $f = (f_1, \ldots, f_n)$ is a smooth vectorfield such that $\sum_k x_k f_k(x) < 0$ for all $x \neq 0$. Show that $x(t) \to 0$ as $t \to \infty$ for all solutions of the system (and all initial data $x(0)$).

3. Let

$$f(x, y) = \begin{pmatrix} ax - bxy - ex^2 \\ -cy + dxy - fy^2 \end{pmatrix},$$

where $a, b, c, d, e, f > 0$. Show that the system $(x', y') = f(x, y)$ has no closed orbit in the first quadrant. (Hint: Show that the divergence of $\frac{1}{xy} f(x, y)$ is non-zero.)

4. Let Ω be an open subset of $\mathbb{R}^n$. Suppose $u \in C^2(\bar{\Omega})$ is a solution of the equation $\Delta u = u^3$ with the property that $|\nabla u(x)| \leq 1$ for each $x \in \partial\Omega$. Prove that $|\nabla u(x)| \leq 1$ for all $x \in \Omega$.

5. Let Ω be an open bounded subset of $\mathbb{R}^n$, and assume $u(x, t) \geq 0$ is a function in $C^2(\bar{\Omega} \times [0, \infty)$ which solves the heat equation with heat loss due to radiation

$$u_t - \Delta u = -u^4$$

with the boundary condition $u = 0$ on $\partial\Omega$. Prove that we can find a constant C such that

$$E(1) = \int_\Omega u^2(x, 1)\, dx \leq C$$

regardless of the initial value $u(x, 0)$.

6. Solve the Cauchy problem

$$x\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -xy$$

$$u = 5 \quad \text{on} \quad xy = 1, x, y > 0$$