Topics for the Graduate Exam in Applied Probability (505a)

Most of the following topics are normally covered in the course Math 505a.

This is a one hour exam.

Sample spaces and probability distributions

Equally likely outcomes, principles of counting, permutations, combinations
Principle of inclusion/exclusion
Conditional probability
Independence
Random variables, distributions, joint distributions (continuous and discrete)
Expectation, moments
Indicators
Covariance, correlation, covariance matrices
Conditional distributions and densities, conditional expectation, conditional variance
Laws of large numbers
CLT
Specific distributions–uniform, normal, exponential, gamma, binomial, Poisson, negative binomial, geometric
Multivariate normal distribution
Generating functions of integer-valued r.v.'s and of general sequences
Characteristic functions (excluding continuity theorem, Bochner's theorem)
Moment generating functions
Branching processes
Simple random walk, reflection principle, Gamblers Ruin
Borel-Cantelli Lemma
Convolution
Modes of convergence (taken individually)

THE FOLLOWING WILL EXPLICITLY NOT BE ON THE EXAM:
Relations among modes of convergence (a.s., in probability, in distribution, in L^p), that is, which ones imply which other ones
Continuity theorem and Bochner's theorem, for characteristic functions

References:

G.R. Grimmett and D.R. Stirzaker, Probability and Random Processes
S. Ross, A First Course in Probability

505a Qualifying exam. November 17, 1999

Do problems 1,2, and **one more problem, chosen from 3,4,5, or 6.**

1.) Let $X, X_1, X_2, \cdots$ be iid with $\mathbb{P}(X > t) = e^{-t}$ for $t > 0$. Let $S_n = X_1 + \cdots + X_n$.

a) Simplify $\mathbb{E}e^{\beta X}$ for $\beta \in (-\infty, 1)$. [You may calculate formally, without rigorous justification.]

b) Simplify $\mathbb{E}X^k$, for $k = 1, 2, 3, \ldots$.

[STRATEGY for c) and d) together: these are the special cases $n = 1, 2$ of a general fact about the order statistics of n independent uniforms constructed from $n+1$ independent exponentials, and you may do the general case once, in place of c) and d). But if d) is too hard, be sure to do c) anyway.]

c) Show that $X_1/(X_1 + X_2)$ is uniformly distributed on [0,1].

d) Let U, V be independent uniform on [0,1]. Let $A = \min(U, V)$ and $B = \max(U, V)$. Show that

$$(A, B) \text{ and } \left(\frac{X_1}{S_3}, \frac{X_1 + X_2}{S_3}\right)$$

have the same joint distribution.

2.) Let $p \in (0, 1)$, let $X_1, X_2, \ldots$ be iid with $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = 0) = 1 - p$, and let $S_n = X_1 + \cdots + X_n$.

a) Simplify a1) $\mathbb{E}S_n$, a2) $\text{VAR}(S_n)$, a3) $\mathbb{E}z^{S_n}$.

b) Let $\lambda > 0$, let N be Poisson with parameter λ, with N and S_n independent for every n. Let $Y = S_N, Z = N - Y$ so that $Y + Z = N$. Show that Y and Z are independent.

c) Let $\lambda, p_1, \ldots, p_r > 0$ with $p_1 + \cdots + p_r = 1$. Let Z_i be Poisson with $\mathbb{E}Z_i = \lambda p_i$ and $Z_1, Z_2, \ldots, Z_r$ independent. Let $N = Z_1 + \cdots + Z_r$. Show that the distribution of the vector $\mathbf{Z} := (Z_1, \ldots, Z_r)$, conditional on the event $N = n$, is multinomial with parameters $(n; p_1, \ldots, p_r)$, for each $n = 0, 1, 2, \ldots$.

d.) For N Poisson with parameter λ and fixed $a > 1$ calculate

$$\lim \frac{\mathbb{P}(N \geq k)}{\mathbb{P}(N = k)},$$

with the limit taken for $\lambda \to \infty, k \to \infty, k/\lambda \to a$. [HINT: geometric series.]

3.) Let π be a random permutation of $\{1, 2, \ldots, n\}$, with all $n!$ possibilities equally likely. Let W be the number of fixed points of π, i.e. $W := \sum_1^n X_i$, where X_i is the indicator of the event that $\pi_i = i$.

a) Compute $\mathbb{E}W$ and $\mathbb{E}W^2$ and simplify, exactly.

b) Show that $\mathbb{P}(W = 0) \to e^{-1}$ as $n \to \infty$.

c) Extend b) by showing that for $k = 0, 1, 2, ...$, $\mathbb{P}(W = j) \to e^{-1}1^j/j!$ as $n \to \infty$. [HINT: you may use the version of inclusion-exclusion known as Waring's formula, which has the form $\mathbb{P}(W = j) = \sum_{k \geq 0}(-1)^k \binom{j+k}{k} s_{j+k}$.]

4.) n people each roll one fair die. For each (unordered) pair of people that get the same number of spots, that number of spots is scored, with S for the total score achieved among the $\binom{n}{2}$ pairs of people. For example, if there are $n = 10$ people, and they roll 1,2,2,2,3,4,4,4,4,6 then $S = 2+2+2+4+4+4+4+4+4$ since there are three pairs of people matching 2 and six = $\binom{4}{2}$ pairs of people scoring 4. [HINT: Consider S as the sum of $\binom{n}{2}$ random variables $S_{i,j}$, where $S_{i,j}$ is k if persons i and j both roll k, and zero otherwise.]

a) Simplify $\mathbb{E}S$.

b) Simplify $\mathbb{E}S^2$.

5.) Let $S_0, S_1, S_2, \ldots$ be simple symmetric random walk, i.e. $\mathbb{P}(S_i - S_{i-1} = 1) = \mathbb{P}(S_i - S_{i-1} = -1) = 1/2$, with independent increments. Let $T = \min n > 0 : S_n = 0$ be the hitting time to zero. Write P_a for probabilities for the walk starting with $S_0 = a$.

a) What does the reflection principle say about $\mathbb{P}_a(S_n = i, T \leq n)$, for $a > 0$, and $i, n \geq 0$?

b) What does the reflection principle say about $\mathbb{P}_a(S_n \geq i, T > n)$, for $a > 0$, and $i, n \geq 0$? [Hint: telescoping series]

c) For fixed $a > 0$, give asymptotics for $\mathbb{P}_a(T > n)$ as $n \to \infty$. [HINT: Stirling's formula is that $n! \sim \sqrt{2\pi n}(n/e)^2$.]

d) Simplify, for fixed $a > 0$, $\lim_{n\to\infty} \mathbb{P}_{a+1}(T > n)/\mathbb{P}_a(T > n)$.

6. Let $X, X_1, X_2, \ldots$ be iid standard normal. Let M_n be the maximum of $X_1, \ldots, X_n$. Let Y have the extreme value distribution, with $\mathbb{P}(L < c) = e^{-e^{-c}}$ for $c \in (-\infty, \infty)$. You will have established a distributional limit of the form $(M_n - a_n)/b_n \Rightarrow L$ by showing that, for $c \in (-\infty, \infty)$

$$\mathbb{P}\left(M_n < \sqrt{2\log n}\left(1 + \frac{c - \log(\sqrt{2\log n}\ \sqrt{2\pi})}{2\log n}\right)\right) \to e^{-e^{-c}}$$

You may use, without proof, the fact that $1 - \Phi(x) \sim \phi(x)/x$ as $x \to \infty$, where $\Phi(x) := \mathbb{P}(X \leq x)$ and $\phi = \Phi'$ is the standard normal density. [HINT: recall that $(1 - a_n)^n \to e^{-t}$ if $na_n \to t$; use this twice.]

Math 505a Exam Portion. Spring 2001

Problem 1.

(i) Each person in a group of n individuals is assigned a different number between 1 and n. After that, each person randomly picks a number between 1 and n from a box. What is the expected value of the number of persons who pick the same number as the one originally assigned to them?

(ii) Consider continuing this game as follows: Suppose that when a person picks the same number as the one originally assigned to them, they stop playing and take the number with them. Those who have not picked their assigned number, try again in the next round, with the remaining numbers in the box. Show that the expected number of rounds necessary for everyone to pick their assigned number is equal to n.

Problem 2. Let X_1, X_2 be normal random variables with mean zero. Suppose $Var(X_1) = 1$ and $Var(X_2) = 2$. Suppose that $X_2 - X_1$ is independent of X_1. Denote, for a given $y > 0$, $\sigma > 0$,

$$Y = ye^{\sigma X_2 - \sigma^2}.$$

a) Compute $E[Y]$.

b) Given $K > 0$, show that

$$E[Y\mathbf{1}_{\{Y>K\}}] = y\Phi\left(\frac{\log(y/K) + \sigma^2}{\sigma\sqrt{2}}\right),$$

where Φ is the cumulative standard normal distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$. Here, $\mathbf{1}_A$ is one if A occurs, and is zero if it does not occur.

c) Find a similar expression for the conditional expectation $E[Y\mathbf{1}_{\{Y>K\}}|X_1]$.

Math 505 Exam

Problem 1: Let $X_0, X_1, \ldots, X_n$ be iid variables with continuous distribution. Let

$$N = \inf\{n \geq 1 : X_n > X_0\}$$

a) Find the conditional distribution of N given X_0,

$$P(N = n|X_0 = x).$$

b) Find the (conditional) mean $E(N|X_0)$ of this distribution.

c) Compute $P(N = n)$ using

$$P(N = n) = E\{P(N = n|X_0)\}$$

d) Compute EN using

$$EN = E\{E(N|X_0)\}$$

e) Interpret the results of c) and d)

Problem 2. Let Y and U be two independent random variables with $Y \sim \mathcal{N}(0,1)$ and $P(U=1) = P(U=-1) = 1/2$. Let $Z = UY$. Show that

a) $Z \sim \mathcal{N}(0,1)$;

b) Y and Z are uncorrelated.

c) Y and Z are **not** independent.

Problem 3. Let S_n be simple symmetric random walk. a) Show that

$$P(S_1 S_2 \ldots S_{2n} \neq 0) = P(S_{2n} = 0) \quad n \geq 1.$$

You may use any method of your choice (e.g. generating functions, one to one path correspondences, etc.)

b) Compute the probability in a).

MATH 505A QUALIFYING EXAM
FEBRUARY 12, 2002

You should try at least 3 problems; you may try all 4.

(1) Let $X_1, X_2, ..$ be iid with characteristic function $\varphi(t) = e^{-|t|^\alpha}$, where $0 < \alpha < 2$.

(a) Show that $\frac{X_1+..+X_n}{n^{1/\alpha}}$ has the same distribution as X_1.

(b) Show that $\text{var}(X_1) = \infty$. HINT: Use (a).

(c) Suppose $\alpha < 1$. Show that the weak law of large numbers does not hold, that is, there is no constant μ such that $\frac{X_1+..+X_n}{n} \to \mu$ in probability.

(2) The Geophysics building at the University of Northern California is scheduled to be seismically reinforced. The reinforcement will occur at a random time uniformly distributed in the next 3 years. Suppose that during any fixed time interval of length t, the number of major earthquakes is Poisson with mean λt. Find the probability that no major earthquake occurs before the reinforcement of the Geophysics building.

(3) A worm farm operates as follows. Let Z_n be the number of worms at the end of month n, with $Z_0 = 1$. Each month, each worm present at the start of the month dies and is replaced by a Binomial(2,p) number of offspring; these numbers of offspring are independent from one worm to another. In addition, with probability r the worm farmer buys one new worm during the month and adds it to the farm, independently of what his current worms are doing. (In other words, he adds a Bernoulli(r) number of worms.)

(a) Let $G_n(s)$ be the generating function of Z_n. Find an equation relating G_{n+1} to G_n.

(b) Suppose $p = r$. Find an explicit formula for $G_2(s)$.

(4) 8 people, including 4 members of the Smith family and 4 members of the Jones family, divide themselves at random (meaning all outcomes are equally likely) into 4 pairs of partners, to play chess. Let N be the number of Smiths whose partners are also Smiths. Find the mean and variance of N. HINT: One approach is to use indicators.

Math 505a Qualifying Exam
Fall 2002

You should try at least 3 problems; you may try all 4.

Problem 1.

An inefficient secretary places n different letters into n differently addressed envelopes at random.

(i) Find the expected number of letters arriving at the proper destination.

(ii) Find the probability that at least one letter will arrive at the proper destination.

Problem 2. Let a_n be the probability that n Bernoulli trials with success probability p result in an even number of successes.

a) Find a relation between a_n and a_{n-1}.

b) Use (a) to calculate a_n.

Problem 3.

If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with mean 1, show that X and Y, defined by

$$X = \sqrt{2Z} \cos U, \quad Y = \sqrt{2Z} \sin U$$

are independent standard normal random variables.

Problem 4.

Let A_n, $n = 1, 2, \dots$, be a sequence of events (not necessarily independent) such that

a) $\sum\limits_{n=1}^{\infty} P(A_n) = \infty$ and

b) $P(A_n \cap A_m) \leq P(A_n)P(A_m) \ \ \forall\, n \neq m$.

True or false: $P\{A_n \text{ i.o.}\} = 1$. Explain. HINT: Consider the mean and variance of $\sum_{i=1}^{n} I_{A_i}$, where I_A denotes the indicator function of the event A. Also, "i.o." means "infinitely often."

Qualifying Exam 505a. February, 2003

1. Let X and Y be independent binomial with parameters N, p and M, p, respectively.

(i) Find the distribution of $X+Y$,

(ii) Find the conditional distribution of X given $X+Y$.

2. In a town of $N+1$ inhabitants a person tells a rumor to a second person, who in turn repeats it to a third person , and so on. At each step the recipient of the rumor is chosen at random from the N inhabitants available.

(i) Find the probability that the rumor will be told (transferred to a person) n times without, (a) returning to the originator, (b) being repeated to any person;

(ii) The rumor mongers constitute $100p\%$ of the population of a large town, which is to say that n, the number of times the rumor was told, is equal to $p(N+1)$ where $N+1$ is the total population of the town. In problem (i), find the limit of the probability in question as $N \longrightarrow \infty$.

3. (i) Let X_1 and X_2 be random variables with joint density function

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}\left[1 + x_1 x_2 (x_1^2 - x_2^2)\right] & \text{if } |x_1| \le 1, |x_2| \le 1, \\ 0 \text{ otherwise.} \end{cases}$$

and $Z = X_1 + X_2$. Prove that the characteristic functions of X_1, X_2, and Z verify $\phi_Z(t) = \phi_{X_1}(t)\phi_{X_2}(t)$.

(ii). Are X_1 and X_2 independent? Prove.

HINT: Take advantage of symmetries to avoid excessive calculation.

4. (i) Show that for $t > 0$ and $x \in \mathbf{R}$, for every r.v. X, $P(X \ge x) \le e^{-tx} E e^{tX}$.

(ii) Let $X_1, X_2, ...$be i.i.d r.v.'s with the distribution function $F_{X_1}(x) = 1 - e^{-x}, x \ge 0$; and $S_n = X_1 + ... + X_n$.Show that

$$P\left(\frac{S_n}{n} > 1 + \varepsilon\right) \le e^{-[\varepsilon - \log(1+\varepsilon)]n}.$$

Hint: Use (i) to solve (ii).

Math 505a Qualifying Exam Problems
Fall 2003

Problem 1.

Let X and Y be discrete random variables taking values $0, 1, 2, ...$ Assume that the joint probability generating function

$$G_{X,Y}(t,s) = \frac{(1 - [p_1 + p_2])^n}{(1 - [p_1 s + p_2 t])^n}$$

where $p_1 + p_2 \leq 1$ and n is a positive integer.

a. Find the marginal mass functions of X and Y.

b. Find the marginal mass function of $X + Y$.

c. Find the conditional probability generating function $G_{X|Y}(s|i) = E\left[s^X | Y = i\right]$

Problem 2.

Let $X_1, X_2, ...$ be iid with distribution function $F(x) = \sqrt{x}$ for $0 \leq x \leq 1$, and let $M_n = \min\{X_1, .., X_n\}$. Show that $n^2 M_n$ converges in distribution, and find the limiting distribution function.

Problem 3.

Consider gambler's ruin with fair bets. Player A starts with k dollars and player B with $N - k$ dollars. Each player wins each round with probability 1/2, gaining \$1 from his opponent. The game ends when one player (the winner of the game) has all N dollars. Let W_k be the event that A wins the game, let R_k be the number of rounds played in the game, and for $1 \leq k \leq N$ let $a_k = E(R_k|W_k)$. It is standard, and you may take as given, that $P(W_k) = k/N$ for all $0 \leq k \leq N$.

(a) Show that

$$(k+1)a_{k+1} - 2ka_k + (k-1)a_{k-1} = -2k, \quad 1 \leq k \leq n. \qquad (1)$$

Note this requires that you choose a definition for $(k-1)a_{k-1}$ when $k = 1$.

(b) The general solution of (1) has form $a_k = b + ck + dk^2$, where b, c are arbitrary constants and d is a specific constant (you may take this fact as given.) Use this to find $E(R_k|W_k)$ explicitly.

Problem 4.

Let X and Y be independent normal random variables with zero means, and unit variances.

a. Prove that $U = (X+Y)/\sqrt{2}$ and $V = (X-Y)/\sqrt{2}$ are independent and Gaussian with zero means and unit variances.

b. Prove that $P(X+Y \leq z | X > 0, Y > 0) = P\left(|U| \leq z/\sqrt{2}, |V| \leq z/\sqrt{2}\right)$

MATH 505a QUALIFYING EXAM
January 27, 2004

You should try at least three problems; you may try all four.

Problem 1. Consider four points, 1,2,3 and 4. For each two points, there is a link between them with probability p, and no link with probability $1-p$, independently of other points. What is the probability that there is a path of connected links between points 1 and 2?

Problem 2. Let X be a non-negative random variable with finite expectation. Show that

$$\sum_{i=1}^{\infty} P(X \geq i) \leq E[X] < 1 + \sum_{i=1}^{\infty} P(X \geq i).$$

Problem 3. Let X and Y be random variables with means equal to zero, variances equal to one, and correlation coefficient ρ. Show that

$$\left(E[\sqrt{\max(X^2, Y^2)}]\right)^2 \leq 1 + \sqrt{1-\rho^2} \ .$$

Hint: First show that $\max(x, y) = (x + y + |x - y|)/2$.

Problem 4. Let $\phi(t)$ be a characteristic function of a random variable X.
a) Assume $\phi(t) = e^{iat}, t \in (-\infty, \infty)$. Show that $\mathbf{P}$-a.s. $X = a$.
b) Assume that $\phi(2\pi) = 1$. Show that

$$\sum_{k=-\infty}^{\infty} P(X = k) = 1.$$

MATH 505a QUALIFYING EXAM
September 20, 2004

You should try at least three problems; you may try all four.

1. There are three coins that show heads with probability 2/3, and tails otherwise. The first coin counts 8 points for a head and 3 for a tail, the second counts 5 points for both head and tail, and the third counts 4 points for a head and 10 for a tail. You and your opponent choose a coin, and you cannot choose the same coin. Each of you tosses your coin and the person with the larger score wins. Would you prefer to be the first to pick a coin, or the second?

2. A coin with probability p for heads is tossed n times. Find the expected number of "runs" and the variance of the number of "runs", where "run" is a sequence of identical outcomes. For example, the sequence $TTTHTHH$ has four runs, TTT, H, T and HH.

3. Let X_n be a sequence of independent identical distributed random variables with distribution function F such that $F(x) < 1$ for all x. Let $Y(y) = \min\{k : X_k > y\}$. Find the probability

$$P(Y(y) \leq E[Y(y)]).$$

Can you also show that the limit of this probability when $y \to \infty$ is $1 - 1/e$?

4. Let $X_1, X_2, \dots, X_n$ be independent identically distributed normal random variables with mean μ and variance σ^2. Consider

$$S_n = X_1 + \cdots + X_n.$$

a) Find the moment generating function of S_n.
b) Find $E(e^{S_n})$.
c) What is the moment generating function of $(S_n - n\mu)/\sqrt{n}\sigma$?

MATH 505a **QUALIFYING EXAM** **February 03, 2005**

Last Name: ________________ **First Name:**________________

ID#: ________________ **Signature:**________________

You should try at least three problems; you may try all four.

1. Let $X_1, X_2, \ldots, X_n$ be a sample of independent, identically distributed random variables with density f and distribution function F. Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the ordered sequence of those variables. For $i < j$ find the joint density of $(X_{(i)}, X_{(j)})$.

2. Let N be a Poisson random variable with parameter λ. Let $Y = \sum_{i=1}^{N} X_i$, where X_i are independent, identically distributed, non-negative integer valued random variables with finite mean. Show that for any function g (such that the expectations exist) we have

$$E[Yg(Y)] = \lambda E[X_0 g(Y + X_0)].$$

3. A stick is broken in two pieces, uniformly at random. Let X denote the ratio of the lengths of the shorter to the longer piece. Find the mean and the variance of X.

4. The number of the electrons that hit the plate is Poisson with parameter $\lambda_1 = 2$. Every impact produces independently a number of secondary electrons that is Poisson with parameter $\lambda_2 = 1$. a) Find the moment generating function of the total number of secondary electrons; b) Find the variance of that number.

MATH 505a **QUALIFYING EXAM** **September 13, 2005**

Last Name: ____________________ **First Name:** ____________________

ID#: ____________________ **Signature:** ____________________

1. A building has $k > 1$ floors above the street level.

Some number $m > 1$ of people enter at street level and board an elevator. They choose floors independently, uniformly at random. Let X_i be the indicator of the event that at least one person selects floor i, so that $S = X_1 + \ldots + X_k$ is the number of stops the elevator must make.

a) Find the constant $a = \mathbf{E}X_1$ as a function of m, k.

b) Find the constant $b = \mathbf{E}X_1X_2$ as a function of m, k.

c) Find $c = \text{Cov}(X_1, X_2)$ in terms of a, b.

d) Use the method of indicators to find $\mathbf{E}S$,

e) Use the variance-covariance expansion to find $\text{Var}(S)$ in terms of a, b, m.

2. Let $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ be two independent simple random walks on $\mathbb{Z}$ starting at zero.

a) Prove that the sequence of ordered pairs $(Z_n = (X_n', Y_n'), n \geq 0)$, where

$$X_n' = \frac{X_n + Y_n}{2} \text{ and } Y_n' = \frac{X_n - Y_n}{2},$$

is a simple random walk on $\mathbb{Z}^2$.

b) For a standard basis vector $\mathbf{e}_i$ in $\mathbf{R}^2$, find $\mathbf{P}(Z_1 = -\mathbf{e}_i)$.

c) Find $\mathbf{P}(Z_n = 0)$.

d) Find the asymptotic of $\mathbf{P}(Z_{2n}' = 0)$ as $n \longrightarrow \infty$.

Hint: Use Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

3. Let X have the binomial distribution $\text{Bin}(n, U)$, where U is uniform on $(0, 1)$. Show that X is uniformly distributed on $\{0, 1, \ldots, n\}$.

Hint. Compute the generating function of X.

4. The n randomly chosen real numbers $a_1, \ldots, a_n$ are rounded off to the nearest hundredth $a_1 + X_1, \ldots a_n + X_n$, where the round-off errors $X_1, \ldots, X_n$ are assumed to be independent and uniformly distributed on $[-\frac{1}{2}10^{-2}, \frac{1}{2}10^{-2}]$.

Use the CLT to find a number $\lambda > 0$ (depending upon n) such that

$$\mathbf{P}(\sum_{i=1}^{n} |X_i| < \lambda) \approx 0.99.$$

STANDARD NORMAL DISTRIBUTION FUNCTION

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt, \qquad x \geq 0.$$

For $x < 0$, use the relation $\Phi(x) = 1 - \Phi(-x)$.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

MATH 505a **QUALIFYING EXAM** **February 1, 2006**

Last Name: ______________ **First Name:** ______________

ID#: ______________ **Signature:** ______________

1. A *run* in a sequence of coin tosses is a maximal subsequence of consecutive tosses all having the same outcome; for example HHHTHHHTTH has 5 runs. A biased coin, with $p = \mathbf{P}(\text{heads}) \in (0,1)$, is tossed n times. Write $q = 1 - p$. Let R_n be the number of runs in the first n tosses. Find exact formulas for

a) $\mu_n = \mathbf{E}R_n$ and

b) $\sigma_n^2 = \text{Var}(R_n)$.

HINT: pay careful attention to boundary effects-what happens at the start and end of the sequence of n tosses. Note that $\mu_1 = 1, \sigma_1^2 = 0$, and use this as a check on your answers. Note also that $\mathbf{P}(R_2 = 1) = p^2 + q^2$, $\mathbf{P}(R_2 = 2) = 2pq$, so $\mu + 2 = 1 + 2pq$.

c) For the special case $p = 1/2$, the distribution of $R_n - 1$ is very well known distribution (e.g. Binomial, Poisson, Hypergeometric, Geometric, etc) NAME the distribution AND its parameter(s).

2. Assume the vector $\mathbf{X} = (X_1, \ldots, X_N)$ has a multivariate normal distribution $N(\mu, \mathbf{V})$, where μ is the vector of expected values and $\mathbf{V}$ is the covariance matrix. Let $c_1, \ldots, c_N$ be constants.

Find the distribution of $Y = \sum_{i=1}^{N} c_i X_i$.

3. Let $S_n = X_1 + \cdots + X_n, n \leq 1$, be a random walk, where $EX_k = \mu$ and $Var(X_k) = \sigma^2$, $0 < \sigma^2 < \infty$.

(a) Find the covariance $Cov(S_n, S_m)$ and the correlation coefficient $\rho(S_n, S_m)$ of S_n and S_m, $m \neq n$.

(b) Assume $n > m$. Find $\lim_{n\to\infty} Cov(S_n, S_m)$ and $\lim_{n\to\infty} \rho(S_n, S_m)$. Does $\lim_{n\to\infty} Cov(S_n, S_m)$ depend on the distribution of the increments? Does $\lim_{n\to\infty} \rho(S_n, S_m)$ depend on the distribution of the increments?

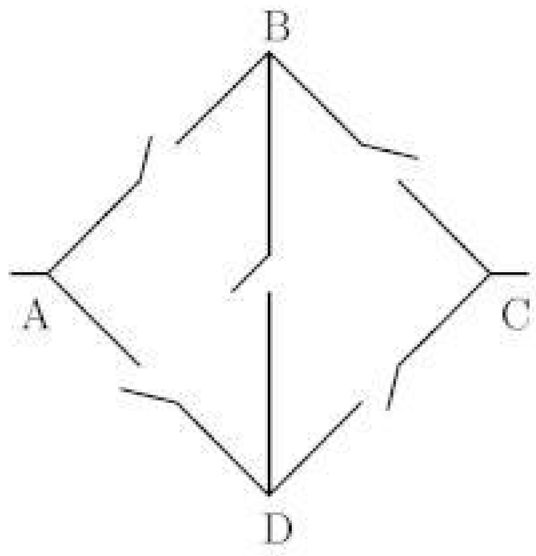


FIGURE 1. A random connection

Fall 2006 Qualifying exam, Math 505a

Do all three problems, attempt all parts

Problem 1. Let X, Y, and Z be independent standard normal random variables.

(a) Show that $X^2 + Y^2$ and $\frac{X}{\sqrt{X^2+Y^2}}$ are independent.

(b) Show that

$$\frac{X + YZ}{\sqrt{1 + Z^2}}$$

is standard normal (Hint: condition on Z).

Problem 2. On Figure 1, each of the five connections can be open or closed independently of other connections. The probability to have a specific connection closed is p.

(a) Find the probability that there is a path of closed connections from A to C.

(b) Find the conditional probability that the connection along the diagonal BD is closed given that there is a path of closed connections from A to C.

Problem 3. Let S_n a random walk on $\mathbb{Z}$, with $S_0 = 0$. Let $\tau_0 = \inf\{n > 0 \colon S_n = 0\}$, the hitting time of 0.

(a) Show that

$$1 = \sum_{m=0}^{n} P_0(S_{n-m} = 0)\, P_0(\tau_0 > m).$$

(Hint: Condition according to the last time, that the chain will visit 0, before time n.)

(b) Assume further that S_n is *simple* random walk, that is, steps are plus one or minus one with probability one-half each. Assume also that n is even. The first term in the sum, indexed by $m = 0$, is simply $P(S_n = 0)$. Give a simple expression a_n which is asymptotic to this, that is, such that the ratio $a_n/P(S_n = 0)$ is close to 1 for large even n.

c) Continuing (b), the last term in the sum, indexed by $m = n$, is simply $P(\tau_0 > n)$. Give a simple expression b_n which is asymptotic to this.

MATH 505a QUALIFYING EXAM January 31, 2007

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) n balls are placed into d boxes at random, with all n^d possibilities equally likely. Assume $d > 8$. Let X be the number of empty boxes. Let D be the event that no box receives more than 1 ball. Let A be the event that boxes 1 and 2 are both empty, B be the event that boxes 3,4,5 are all empty, and C be the event that boxes 6,7,8 are all empty.

a) Calculate and simplify: $\mathbb{E}\,X =$ ____________________

b) Calculate and simplify: $\mathrm{Var}\,\mathrm{X} =$ ____________________

c) $\mathbb{P}(A \cup B \cup C) =$ ____________________

d) If both $n, d \to \infty$ together, what relation must they satisfy in order to have $\mathbb{P}(D) \to .1$?

2.) Suppose Z is Poisson with $\mathbb{E}\,Z = \lambda < 1$. Let $X = 2^Z, Y = Z!$. Compute and simplify each of the following:

a) $\mathbb{E}\,Z^2 =$ ____________________
b) $\mathbb{E}\,Z^3 =$ ____________________
c) $\mathbb{E}\,Y =$ ____________________
d) $\mathbb{E}\,X =$ ____________________
e) $\mathrm{Var}\,\mathrm{X} =$ ____________________

3) Suppose that X is a sum of indicator random variables, with $\mu = \mathbb{E}\,X = 10, \sigma^2 = \mathrm{Var}\,\mathrm{X} = 7$. Let A be the event $\{X > 0\}$.

a) State Chebyshev's inequality, involving the variance and the distance to the mean.
b) Apply Chebyshev's inequality to get a lower bound on $\mathbb{P}(A)$.
c) State the Cauchy-Schwarz for $(\mathbb{E}\,(XY))^2$.
d) Apply Cauchy-Schwarz, with $Y = 1(X > 0)$, the indicator that X is strictly positive, to get a lower bound on $\mathbb{P}(A)$.

MATH 505a GRADUATE EXAM

FALL 2007

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) (Daniel Bernoulli, 1786) Of the $2n$ people in a collection of n couples, exactly m die, with all $\binom{2n}{m}$ possibilities equally likely. Find the expected number of surviving couples.

(2) Let X be a standard normal random variable, and let Y be independent of X with $P(Y = 1) = P(Y = -1) = 1/2$. Answer the following questions and *justify your answers*:

(a) Is the random variable $Z = XY$ normally distributed?

(b) Do X and Z have a nonzero correlation?

(c) Does (X, Z) have a bivariate normal distribution?

(3) A sequence of mean-0 random variables $(X_n)_{n\in\mathbb{N}}$ is called *weakly stationary* if there is a function ϕ such that

$$E[X_iX_j] = \phi(|j - i|) < \infty \quad \text{for all } i, j,$$

in other words this expected value only depends on the difference $|j - i|$. Suppose that for some such sequence, we have $\phi(k) \to 0$ as $k \to \infty$. Show that the weak law of large numbers is valid, that is, $\frac{X_1+\cdots X_n}{n}$ converges to 0 in probability.

(4) Consider a branching process with immigration: each generation is supplemented by an "immigrant" with probability p. This means that the size Z_n of the nth generation satisfies

$$Z_{n+1} = I_{n+1} + \sum_{i=1}^{Z_n} X_i,$$

where $I_{n+1} = 1$ with probability p, 0 with probability $1 - p$, and the family sizes X_i are i.i.d. with generating function $G(s)$. We assume Z_n, I_{n+1} and $\{X_i\}$ are independent. Let $G_n(s)$ be the generating function of Z_n and let $\mu_n = EZ_n$.

(a) Show that $G_{n+1}(s) = (ps + (1 - p))G_n(G(s))$. HINT: Condition on Z_n.

(b) Show that $\mu_{n+1} = p + \mu_n\mu$.

(c) If $\{\mu_n\}$ converges to a finite limit μ_∞, then what is μ_∞, in terms of p and μ?

MATH 505a GRADUATE EXAM

SPRING 2008

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $Y \geq 0$ be a random variable with density f, and let X be another random variable. Assume X and Y have finite variances. Show that $E(XY) = \int_0^\infty E(XI_{[Y \geq t]})\,dt$. HINT: First express $E(XI_{[Y \geq t]})$ as an integral involving $E(X \mid Y = y)$.

(2) Let $\mathbf{X} = (X_1, .., X_m)$ and $\mathbf{Y} = (Y_1, .., Y_m)$ be random vectors with covariance matrices $\Sigma_\mathbf{X}$ and $\Sigma_\mathbf{Y}$.

(a) The *cross-covariance matrix* of $\mathbf{X}$ and $\mathbf{Y}$ is given by $C_{ij} = \text{cov}(X_i, Y_j)$. For vectors $\mathbf{a}, \mathbf{b}$, express $\text{var}(\mathbf{a} \cdot \mathbf{X} + \mathbf{b} \cdot \mathbf{Y})$ in terms of $\mathbf{a}, \mathbf{b}, \Sigma_\mathbf{X}, \Sigma_\mathbf{Y}$ and C.

(b) Suppose that for some vectors $\mathbf{a}, \mathbf{b}$ and some $k \in \mathbb{R}$, we have $\text{var}((\mathbf{a}+u\mathbf{b})\cdot\mathbf{X}) = ku$ for all $u \in \mathbb{R}$. Show that there are constants c_1, c_2 such that $P(\mathbf{a} \cdot \mathbf{X} = c_1) = P(\mathbf{b} \cdot \mathbf{X} = c_2) = 1$, and determine the value of k.

(3) Let U be a standard Cauchy random variable, that is, the density of U is $f_U(x) = \frac{1}{\pi}\frac{1}{1+x^2}, x \in \mathbb{R}$.

(a) Show that U and $1/U$ have the same distribution.

(b) Show that $E|U|^\alpha \geq 1$ for all $0 < \alpha < 1$. HINT: $1 = U \cdot \frac{1}{U}$.

(4) A sequence $X_1 X_2 \ldots X_n$ is said to have a local maximum at 1 if $X_1 > X_2$, a local maximum at i (for $1 < i < n$) if both $X_i > X_{i-1}$ and $X_i > X_{i+1}$, and a local maximum at n if $X_n > X_{n-1}$. Let N be the number of local maxima.

(a) Find the mean and variance for N in each of the following cases:

(i) if $X_1 X_2 \ldots X_n$ is a random permutation of the numbers $1, 2, \ldots, n$, with all $n!$ permutations equally likely;

(ii) if $X_1, X_2, \ldots, X_n$ are chosen independently and uniformly from the integers $\{1, 2, \ldots, q\}$.

(b) Pick one of the two cases (i) or (ii) in (a), and show that $N/n \to 1/3$ in probability as $n \to \infty$.

MATH 505a QUALIFYING EXAM September 23, 2008

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) The number of cars arriving at a McDonald's drive-up window in a given day is a Poisson random variable, N, with parameter λ. The numbers of passengers in these cars are independent random variables, X_i, each equally likely to be one, two, three, or four.

a) Simplify the probability generating function of N, say $G_N(z) = \mathbb{E}\, z^N =$ _____.

b) Simplify the moment generating function of $X = X_i$, say $M_X(t) = \mathbb{E}\, e^{tX} =$ _____.

c) Find the moment generating function of the total number of passengers passing by the drive-up window in a given day. (Hint: $S = \sum_{i=1}^{N} X_i$.)

2.) Consider a lottery with n^2 tickets, of which exactly n win prizes. A person buys $2n$ tickets. Find the following limits; part credit for guessing plausibly, and part credit for a proof.[Hint: the lottery involves drawing without replacement, but a good guess arises by thinking of drawing tickets with replacement.]

a) $\lim_{n\to\infty} \mathbb{P}($ at least one winning ticket$) =$ _____

b) $\lim_{n\to\infty} \mathbb{P}($ exactly 3 winning tickets$) =$ _____

3a) Suppose that S and S' are iid standard exponential, and $r > 0$.
Show that $P(rS < S') = 1/(1+r)$. [Hint: you might answer either by a detailed calculation, or by an informal Poisson process argument.]

3b) Find the covariance of (D_1, D_2, D_3, D_4) where $D_1 = (X_2 - X_1)/2, D_2 = (Y_2 - Y_1)/2, D_3 = X_3 - (X_1 + X_2)/2, D_4 = Y_3 - (Y_1 + Y_2)/2$ and the six coordinates (X_i, Y_i) for $i = 1$ to 3 are iid standard normal.

3c). Suppose that A, B, C are points in the plane, whose six coordinates (X_i, Y_i) for $i = 1$ to 3 are iid standard normal. There is a unique circle having the segment from A to B as a diameter. Show that the probability that C lies inside this circle is 1/4. [Hint: 3a) and 3b) are both useful here. Think about the square of the distance from C to the midpoint of A,B, and about the square of the radius of the circle.]

MATH 505a QUALIFYING EXAM February 5, 2009

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. random variables with a continuous distribution function. Let

$$N = \min\{n \geq 2 : \; X_n > X_{n-1}\}.$$

a) Find $\mathbb{P}(N = n)$ for $n = 2, 3, \ldots$.
b) Find $\mathbb{P}(N \geq n)$ for $n = 1, 2, 3, \ldots$.
c) Find and simplify EN.

2.) A deck (deck #1) of cards has a red cards and b black cards, another deck (deck #2) has c red cards and d black cards. Assume $a, b, c, d \geq 1$. Both decks are well-shuffled. Suppose that you pick f cards ($0 \leq f \leq a + b$) randomly from deck #1 and mix them into deck #2. One card is now selected and removed from the mixed up deck #2.
a) What is the chance that the first card selected is red?
b) Given that the first selected card is red, what is the chance that it originally came from deck #1 ?
c) Are the two events, that the first selected card is red, and that the first selected card originally came from deck #1, independent? [Possible answers include YES, NO, or some relation among the parameters a, b, c, d, f. If you give a relation, please simplify it.]

3.) Suppose that the random variables Y, X, X_1, X_2, ..., are independent, identically distributed, and that $P\{Y = n\} = 2^{-n}$, for $n = 1, 2, \ldots$; and $P\{X \geq t\} = e^{-\pi t}$, for $t > 0$ and $k = 1, 2, \ldots$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Let $Z = S_Y$.

a) Calculate and simplify $\mathbb{E}\, Z$.
b) Simplify the probability generating function $\mathbb{E}\, s^Y$.
c) Simplify the moment generating function $\mathbb{E}\, \exp(\beta X)$.
d) Simplify $\mathbb{E}\, \exp(\beta Z)$.

e) Calculate and simplify $\mathbb{E}\, Z^3$.

[Hint, if you answered a) by some easy method, use this to check your results for d) and e).]

Last Name: ____________________ **First Name:**____________________

1. A drawer contains N pairs of socks; each sock has *precisely* one mate. The $2N$ socks are randomly arranged in the drawer. I choose k socks (randomly) from among the $2N$ socks in the drawer, with $2 \le k \le 2N$. What is the *expected* number of complete pairs in my sample of k socks?

2. A clerk in a gas station is rolling a fair dice while waiting for the customers to come. Suppose that the number of times the dice is rolled between two customers has a Poisson distribution with parameter $\lambda = 5$. Let ξ be the total points (of the dice) the clerk observed right before the next customer comes in. Determine $E\xi$ and $D\xi$ (standard deviation).

3. Let a random variable X be normal $N(\mu, \sigma^2)$ and let the conditional distribution of Y given X be normal $N(a_1 + a_2 X, \sigma_1^2)$.

(a) Find the joint probability density function of X and Y.

(b) Find the marginal distribution of Y and the correlation coefficient of X and Y.

4. Let ξ and η be two random variables, both taking only two values. Show that if they are uncorrelated, then they are independent.

MATH 505a Screening Exam Spring, 2010

Last Name: ____________________ **First Name:**____________________

1. Let (X_1, X_2) be standard bivariate normal random variables with correlation $\rho = \frac{3}{5}$. Let (Y_1, Y_2) denote the midterm exam score and final exam score of a randomly selected student in class. Assume

$$Y_1 = 80 + 3X_1, Y_2 = 75 + 2X_2.$$

If a student got 90 in the midterm exam,

a) what is the conditional expectation and conditional variance of his/her final exam score?

b) What is the conditional probability that he/she got more than 75 in the final exam?

2. Let $\{S_k\}_{k\geq 0}, S_0 = 0$, be a symmetric simple random walk. For an integer $n \geq 1$, let $\tau_n = \min\{k \geq 1 : S_k \notin (-n, n)\}$ be the first time k such that S_k leaves the region $(-n, n), n \geq 1$, where $\tau_n = \infty$ if there is no such k. Find the moment generating function of $S_{\tau_n}, \mathbf{E}S_{\tau_n}$ and $\mathrm{Var}(S_{\tau_n})$.

3. Let $\Theta_1, \Theta_2, \ldots$ be a sequence of independent, identically distributed random variables with the uniform distribution on the interval $(0, 2\pi)$. For $n = 1, 2, \ldots$ define

$$X_n = \sum_{k=1}^{n} \cos\Theta_k, Y_n = \sum_{k=1}^{n} \sin\Theta_k, \text{ and } R_n^2 = X_n^2 + Y_n^2.$$

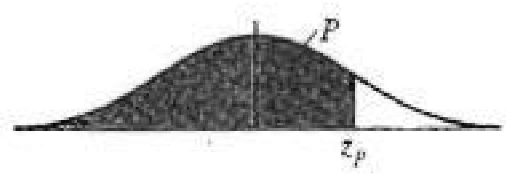
Show that

a) there is a sequence of numbers a_n so that $(a_n X_n, a_n Y_n)$ has a limiting bivariate normal distribution as $n \to \infty$;

b) $\lim_{n\to\infty} \mathbf{P}(R_n^2 \geq n)$ exists.

T A B L E 2

Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	*.00*	*.01*	*.02*	*.03*	*.04*	*.05*	*.06*	*.07*	*.08*	*.09*
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Solve all four problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. Let X_k, $k \geq 1$, be iid random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2}$$

exists in an appropriate sense, and identify the limit.

Problem 2. Fix $p \in (0,1)$ and consider independent Poisson random variables X_k, $k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}.$$

Verify that the sum $\sum_{k=1}^{\infty} kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum_{k=1}^{\infty} kX_k$.

SUGGESTION: compute the generating function for X_k, for kX_k, and for Y.

Problem 3. A coin-making machine produces quarters in such a way that, for each coin, the probability U to turn up heads is uniform on $[0,1]$. A coin pops out of the machine.

(a) Compute the conditional distribution of U given that the coin is flipped once and lands on head.

(b) Compute, either exactly or approximately, the conditional distribution of U given that the coin is flipped 2000 times and lands on head 1500 times.

Problem 4. An ordered vertical stack of n books is on my desk. Every day, I pick one book uniformly at random from the stack and put the book on top of the stack. What is the expected number of days before the books are back to the original order?

COMMENTS: (a) For partial credit, just guess the answer, as a function of n. (b) For more credit, give a heuristic justification. (c) For bonus credit, give a proof.

MATH 505a QUALIFYING EXAM February 2, 2011

Solve all problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, **and write on only one side of the paper**. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. `This is a basic computational problem.`

Let X be normal with zero mean and variance σ^2, let Θ be uniform on $(0, \pi)$, and let a be a real number. Assume X and Θ are independent. Find the density of $Z = X + a\cos(\Theta)$.

Problem 2. `This is an abstract version of the coupon collector problem.`

(a) Suppose that r balls are placed at random into n boxes. Show that if

$$\lim_{n,r\to\infty} ne^{-r/n} = \lambda \in (0, \infty),$$

then, in the same limit, the number of empty boxes has Poisson distribution with mean λ.

(b) Let $X_1, X_2, \ldots$ be iid uniform on $\{1, 2, \ldots, n\}$: for each $k \geq 1$ and $m = 1, \ldots, n$, $P(X_k = m) = 1/n$. Define

$$T_n = \inf\{m : \{X_1, \ldots, X_m\} = \{1, 2, \ldots, n\}\}.$$

That is, T_n is the first time the sample $X_1, \ldots, X_m$ contains all the numbers from 1 to n. Show that

$$\lim_{n\to\infty} P\big(T_n - n\ln(n) \leq nx\big) = \exp(-e^{-x}).$$

Suggestion. Use part (a) and note that $T_n \leq m$ if and only if m balls fill up all n boxes.

Problem 3. `This problem tests your ability to work with indicator random variables and to use the variance-covariance expansion.`

Each member of a group of n players rolls a die (an ordinary fair die with faces number 1 to 6.) For every pair of players who throw the same number, the group scores 1 point. For example, if $n = 10$ and the ordered results of the rolls are 1,2,3,3,3,5,5,5,5,6, then group's score is 3+6=9. In the unlikely event that everyone rolls the same number, the maximal possible score, $\binom{n}{2}$, is achieved.

Find the mean and variance of the total score of the group.

Applied Probability (505A) Graduate Exam

Fall 2011

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. True or false: if A and B are events such that $0 < P(A) < 1$ and $P(B|A) = P(B|A^c)$, then A and B are independent. Justify your answer.

2. Suppose X and Y are independent, each with exponential density e^{-x} for $x > 0$, and let $Z = X - Y$.

(a) Calculate the density of Z.

(b) Calculate the moment generating function of X, the characteristic function of X, and the characteristic function of Z.

3. On the first day of class, the professor observed that there are m men and n women in the class. He says "I will bet even money, that some man-woman pair of students in this class have the same birthday."

Let W denote the number of man-woman pairs which have a birthday in common, so that the professor is betting that $W > 0$. (For example, if Fred, Bob, Mary, Jane, and Linda all have the same birthday, then $W \geq 6$.) You may assume that the birthdays of the $m + n$ students are distributed independently and uniformly over the 365 days of a non-leap year.

(a) Find an exact simple expression for the expectation of W.

(b) Find and simplify the variance of W.

(c) Suppose $m = 10$ and $n = 20$. Name a simple distribution that gives a good approximation for the distribution of W. (There may be more than one acceptable answer; no proof is required.)

(d) With $m = 10$ and $n = 20$, does the professor expect to win or lose money? You can give a heuristic approximation, but your answer should involve some calculation. Recall that $\ln 2 \approx 0.693$.