

Topics for the Graduate Exam in Algebra

Most of the following topics are normally covered in a two semester graduate sequence in algebra (510ab).

This is a two hour exam.

Groups: Review of elementary group theory, isomorphism theorems, group actions, orbits, stabilizers, simplicity of A_n , Sylow's theorems, direct products and direct sums, semi-direct products, Fundamental Theorem of Abelian Groups, solvable groups.

Fields: Relative dimensions, automorphisms, splitting fields, isomorphism extension theorem, separable extensions, Galois correspondence, Fundamental Theorem of Galois Theory, principal element theorem, traces and norms, radical extensions, finite fields, cyclotomic extensions, algebraic closure.

Commutative Algebra: Localization, integral extensions, unique factorization domains, Eisenstein criterion, principal ideal domains, Noetherian rings, Hilbert basis theorem, varieties, Zariski topology, Hilbert Nullstellensatz.

Modules: Irreducible modules, torsion modules, free modules, projective modules, modules over PIDs, chain conditions, tensor products, exact sequences.

Noncommutative Rings: Artinian rings, Jacobson radical, Artin-Wedderburn theorem, Skolem-Noether theorem, division rings, Wedderburn's theorem on finite division rings.

References:

D. Rotman, The Theory of Groups

S. Lang, Algebra

T. Hungerford, Algebra

T.Y. Lam, A first course in non-commutative rings

M. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra

I. Stewart, Galois theory

H.M. Edwards, Galois theory

ALGEBRA QUALIFYING EXAM (MATH 510AB)

SPRING 1992

- (1) Let G be a group with $|G| = 5 \cdot 7^2 \cdot 17$. Determine the possible structures for G .
- (2) Let G be a finitely generated abelian group $G \neq \{1\}$, n a positive integer and let $\varphi_n : G \rightarrow G$ be the homomorphism defined by $\varphi_n(g) = g^n$. Prove
 - (a) G is not divisible, i.e. $\exists n$ such that φ_n is onto.
 - (b) G is finite if and only if $\exists n$ such that φ_n is the trivial map, i.e. $\varphi_n(g) = 1$ for all $g \in G$.
 - (c) G is free abelian group if and only if φ_n is 1-1 for all positive integers n .
 - (d) G is finite if and only if φ_n is an isomorphism for some positive integer $n > 1$.
- (3) Let $k = \mathbb{Q}(\zeta_{15})$, where ζ_{15} is a primitive 15-th root of unity. What is the galois group of k/\mathbb{Q} ? How many subfields does k have? List all subfields (recall that the field $\mathbb{Q}(\zeta_p)$ of p -th roots of unity, p a prime, contains the subfield $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 \pmod{4}$ and $\mathbb{Q}(\sqrt{-p})$ if $p \not\equiv 1 \pmod{4}$).
- (4) Let $F = \mathbb{F}_{p^n}$ be a field of p^n elements. For $1 \leq k \leq n$ set $L_k = \{a \in F : a^{p^k} = a\}$. Show that each L_k is a subfield of F , that $\{L_k : 1 \leq k \leq n\}$ is the set of all subfields of F , and for n greater than 2, $L_i = L_j$ for some $1 \leq i < j \leq n$.
- (5) Let A be a commutative noetherian ring, M a noetherian A -module.
 - (a) Prove that $M \otimes_A A[x]$ is a noetherian $A[x]$ -module.
 - (b) If A is a commutative Noetherian domain with 1, and $0 \neq y \in A$, a nonunit. Show that $y = a_1 a_2 \dots a_k$ with each $a_i \in A$ irreducible.
 - (c) Let $\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring in n variables over the complex numbers. An (irreducible) hypersurface in \mathbb{C}^n is the solution set $Z(f)$ of $f(x) = 0$, f an irreducible polynomial in $\mathbb{C}[X]$. Let $\mathcal{F}(Z(f))$ denote the ring of complex valued polynomial functions on the hypersurface $Z(f)$; i.e. $h \in \mathcal{F}(Z(f))$ if and only if $\exists g \in \mathbb{C}[X]$ such that $h(T) = g(T)$ for all $T \in Z(f)$. Prove that $\mathcal{F}(Z(f)) \cong \mathbb{C}[X]/f(X)\mathbb{C}[X]$.
 - (d) Let A be a finite dimensional semi-simple algebra over \mathbb{C} , and set $M_n(A) =$ ring of $n \times n$ matrices over A . Note that $M_n(M_m(A)) \cong M_{mn}(A)$ and $M_n(A \oplus B) \cong M_n(A) \oplus M_n(B)$.
 - (a) Show that $M_2(A)$ is semi-simple.
 - (b) If $\dim_{\mathbb{C}}(A)$ is prime, show that $M_2(A)$ is not simple.
 - (c) If A is not commutative, there is a $t \in M_2(A)$ with $t^3 \neq 0$ and $t^4 = 0$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 1992

- (1) If G is a group of order $2^3 \cdot 19 \cdot 23$ show that G has a normal subgroup of order $4 \cdot 19 \cdot 23$ and the center of G contains an element of order 2.
- (2) Let $T : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ be a group homomorphism.
 - (a) If T is onto, show that $\mathbb{Z}^n \cong \text{Ker } T \oplus \mathbb{Z}^m$.
 - (b) Prove that T is injective if and only if $m \geq n$ and $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/T(\mathbb{Z}^n)) = m - n$.
- (3) Let F be a finite Galois extension of the field k , A subfield $k \subset L \subset F$ is abelian if L is a Galois extension of k and $\text{Gal}(L/k)$ is abelian.
 - (a) Prove there is a unique maximal abelian subfield of F .
 - (b) Prove that if L, J are abelian extensions of k then their composite is an abelian extension of k .
- (4) Let L be the splitting field of $x^{98} - 1$ over \mathbb{Q} . Find $[L : \mathbb{Q}]$, $|\text{Gal}(L/\mathbb{Q})|$, the structure of $\text{Gal}(L/\mathbb{Q})$, the number of subfields of L and the subfields which are normal over \mathbb{Q} .
- (5) (a) Let $M_2 \xrightarrow{g} M_3 \xrightarrow{h} M_4 \rightarrow 0$ be an exact sequence of A -modules, A a ring and $f : M_1 \rightarrow M_2$ an A -module homomorphism. Prove that $M_1 \xrightarrow{g \circ f} M_3 \xrightarrow{h} M_4 \rightarrow 0$ is exact if and only if $f(M_1) + \text{ker } g = M_2$.
 - (b) Show that whenever $A = k[x, y]$, k a field, and I is an ideal in A then there is an exact sequence $A^m \rightarrow A^n \rightarrow I \rightarrow 0$ for some positive integers m, n .
- (6) For an ideal I in $A = \mathbb{C}[x, y, z]$ set $Z_{xy}(I) = \{(a, b) \in \mathbb{C}^2 : f(a, b, z) = 0 \text{ for all } f \in I\}$
 - (a) Prove that I maximal implies Z_{xy} is empty.
 - (b) Prove that $Z_{xy}(I) \times \mathbb{C} = Z(I) = \{(a, b, c) \in \mathbb{C}^3 : f(a, b, c) = 0 \forall f \in I\}$ if and only if $\text{rad } I = JA$, where $J = \{f(x, y) : f(a, b) = 0 \forall (a, b) \in Z_{xy}\}$.
- (7) Describe up to isomorphism all semi-simple \mathbb{C} subalgebras of $M_4(\mathbb{C})$, the ring of 4×4 matrices over \mathbb{C} . (Note that if A, B are \mathbb{C} -algebras and $\alpha \in A, \beta \in B$ have minimal polynomials, f, g respectively then $(\alpha, \beta) \in A \oplus B$ has minimal polynomial $h = \text{lcm}(f, g)$).

ALGEBRA QUALIFYING EXAM (MATH 510AB)

SPRING 1993

- (1) Find up to isomorphism all groups of order $3 \cdot 7 \cdot 19 \cdot 37$.
- (2) Let D be a commutative domain with multiplicative identity 1 and assume that the additive group D is finitely generated. Prove
 - (a) Characteristic $K = 0$ if and only if $(D, +)$ is a free abelian group.
 - (b) If \exists an integer $n > 1$ such that $f : D \rightarrow D$ defined by $x \mapsto nx$ is onto then D is a finite field.
 - (c) If M is a maximal ideal in D then $M \cap i(\mathbb{Z}) = \pi(\mathbb{Z})$ for some prime p , $i(\mathbb{Z}) = \{n \cdot 1, n \in \mathbb{Z}\}$.
- (3) Let $f(x) \in \mathbb{Q}(x)$ be irreducible with $\deg f = n$. Let $M \subset \mathbb{C}$ be a splitting field for $f(x)$ over \mathbb{Q} .
 - (a) Show that if $\text{Gal}(M/\mathbb{Q})$ is abelian then every subfield of M is Galois over \mathbb{Q} .
 - (b) Show that if $\text{Gal}(M/\mathbb{Q})$ is abelian then $[M : \mathbb{Q}] = n$.

Let $f(x) \in \mathbb{F}_{p^n}[x]$ be irreducible of degree t , \mathbb{F}_{p^n} a field with p^n elements, p a prime.

 - (a) Show that $\mathbb{F}_{p^{nt}}$ is a splitting field of f over \mathbb{F}_{p^n} .
 - (b) For $n = 1$, show that $f(x)$ divides $x^{p^m} - x$ if and only if t divides m .
 - (c) How many distinct irreducibles in $\mathbb{F}_2[x]$ have degree 5?
- (4) Let $f_i(x, y) = a_i x^2 + b_i xy + c_i y^2 \in \mathbb{C}[x, y]$, $1 \leq i \leq n$. Show that there exists $(u, v) \in \mathbb{C}^2$ such that $u^2 + v^2 = 1$, but $f_i(u, v) \neq 0 \forall i = 1, \dots, n$.
- (5) Given the linear equation $a_1 X_1 + \dots + a_t X_t = 0$, $a_i \in A = k[x_1, \dots, x_m]$ and k a field, prove that there are solutions $Y_1, \dots, Y_q \in A^t$ such that for each solution Y , there exists $b_1, \dots, b_q \in A$ such that $Y = \sum_{i=1}^q b_i Y_i$. If $A = \mathbb{Z}$, prove that you can take $q = t - 1$.
- (6) Let x denote a fixed non zero vector in \mathbb{C}^3 and A_x denote the ring of matrices $T \in \mathbb{M}_3(\mathbb{C})$ such that $xT = 0$.
 - (a) Prove that $A_{xU} \cong A_x$ for any $U \in \text{GL}_3(\mathbb{C})$, hence $A_x \cong A_y$ for any non zero $y \in \mathbb{C}^3$.
 - (b) Prove that $\{(a_{ij}) \in A_{(1,0,0)} : a_{ij} = 0 \text{ for } j > 1\}$ is nilpotent ideal in $A_{(1,0,0)}$.
 - (c) Prove that the Jacobson radical $J(A_x)$ is not zero and that $A_x/J(A_x) \cong \mathbb{M}_c(\mathbb{C})$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

SPRING 1995

- (1) Let G be a finite group.
 - (a) If $|G| = 2^s \cdot 7$ for $0 \leq s \leq 3$, then G is solvable.
 - (b) Suppose $|G| = 112$ and G is not simple. Show that G is solvable.
 - (c) If $|G| = 112$, show that G is not simple. *Hint:* If G were simple, show that G embeds into S_7 , and then show that G embeds, in fact, into A_7 .
- (2) Suppose that a group G is the direct sum of k cyclic groups, each of prime power order. If H is a subgroup of G containing nontrivial subgroups H_1, \dots, H_s , whose sum is direct, show that $s \leq k$.
- (3) Let $f(x) = (x^3 - 3)(x^2 + 1) \in \mathbb{Q}[x]$. Denote by K the splitting field of $f(x)$ over \mathbb{Q} , and by G the Galois group of K over \mathbb{Q} .
 - (a) Find $|G|$.
 - (b) Show that G has a normal subgroup of order 2.
 - (c) Show that the 3-Sylow subgroup of G is normal. *Hint:* K contains all 12-th roots of unity.
 - (d) Show that G has a central element of order 2.
- (4) Let R be a right Noetherian ring (with 1). Prove that R has a unique maximal nilpotent ideal $P(R)$. Show that the polynomial ring $R[x]$ must also have a unique maximal nilpotent ideal $P(R[x])$, and that $P(R[x]) = P(R)[x]$.
- (5) Let M be a maximal ideal in the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. Show that there are only finitely many maximal ideals in $\mathbb{C}[x_1, \dots, x_n]$ containing M . *Hint:* Show first that for each i there is a polynomial $f_i(y) \in \mathbb{Q}[y]$ such that $f_i(x_i) \in M$.
- (6) Let G be a finite group, and $\mathbb{C}[G]$ its group algebra. Define a bijection $*$: $\mathbb{C}[G] \rightarrow \mathbb{C}[G]$ as follows: For $x = \sum_{g \in G} a_g g \in \mathbb{C}[G]$, set $x^* = \sum_{g \in G} \bar{a}_g g^{-1}$. One calls x symmetric if $x = x^*$.
 - (a) Given $x, y \in \mathbb{C}[G]$, show that $(x^*)^* = x$, and $(xy)^* = y^* x^*$.
 - (b) Given $x \in \mathbb{C}[G]$, show that xx^* is symmetric, and that $xx^* = 0$ if and only if $x = 0$.
 - (c) Show that nonzero symmetric elements are not nilpotent.
 - (d) Assume that $\mathbb{C}[G]$ has no nonzero nilpotent elements. Show that G is abelian.
- (7) Let K be a field extension of $k = \mathbb{F}_{p^r}$ of degree n . Let σ be the automorphism of K given by $\sigma(a) = a^{p^r}$ for $a \in K$.
 - (a) Let x be an element of K such that both $x + \sigma(x)$ and $x\sigma(x)$ belong to k . Show that $[k(x) : k] \leq 2$. Moreover, show that $[k(x) : k] = 2$ if and only if $\sigma(x) \neq x$.
 - (b) Set

$$F = \{x \in K \mid x + \sigma(x), x\sigma(x) \in k\}$$

Show that F is a subfield of K , and that $[F : k] \leq 2$. Moreover, show that $[F : k] = 2$ if and only if $2 \mid n$.

Partial credit is given for partial solutions.

1. Up to isomorphism, determine all groups of order $7^2 \cdot 11^2 \cdot 19$.
2. Let G be a finite Abelian group and recall that the *exponent* of G is the smallest positive integer n so that $g^n = e_G$ for all $g \in G$. Show that G is a cyclic group if and only if the order and exponent of G are equal.
3. An ideal in a commutative ring is called *irreducible* if it cannot be written as an intersection of finitely many properly larger ideals. If $A = \mathbf{Z}[x_1, \dots, x_n]$ is the polynomial ring in n variables over \mathbf{Z} , show that any ideal of A is an intersection of finitely many irreducible ideals.
4. Let K be a splitting field over \mathbf{Q} of the polynomial $x^{11} - 17$. Show that $\text{Gal}(K/\mathbf{Q})$ is isomorphic to the group of matrices $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in \text{GL}(2, \mathbf{Z}/11\mathbf{Z}) \right\}$.
5. Let F be a field and M an irreducible (i.e. simple and nontrivial) $F[x_1, \dots, x_n]$ module.
 - i) If F is algebraically closed, show that $\dim_F M = 1$.
 - ii) For any F show that $\dim_F M$ is finite.
6. Let R be a finite ring in which every element is a sum of nilpotent elements. Show that R is nilpotent. (Hint: what is the trace of a nilpotent element in $M_n(F)$ for F a field?)

Partial credit is given for partial solutions.

1. Let G be a finite group and p a prime number. Let P be a p -Sylow subgroup of G and denote the normalizer of P in G by $N_G(P)$.
 - i) Show that $N_G(P) = N_G(N_G(P))$.
 - ii) If K is a normal subgroup of G and K contains P , show that $G = KN_G(P)$.
 - iii) If no proper subgroup of G is its own normalizer, show that the center of G is not trivial.
2. Up to isomorphism describe all finitely generated Abelian groups which satisfy all of the following properties: i) $G \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^2$; ii) $G \otimes_{\mathbf{Z}} (\mathbf{Z}/7\mathbf{Z}) \cong (\mathbf{Z}/7\mathbf{Z})^3$; and iii) for any prime $p \neq 7$, $G \otimes_{\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z}) \cong (\mathbf{Z}/p\mathbf{Z})^2$.
3. Let R be a left Artinian ring with Jacobson radical $J(R)$. If $R \neq J(R)$ show that R is a left Noetherian ring.
4. Determine if each of the following polynomials is irreducible, and justify your answer.
 - i) $x^{2^n} + 1 \in \mathbf{Q}[x]$.
 - ii) $x_n^n + x_{n-1}^{n-1} + \cdots + x_2^2 + x_1 \in F[x_1, \dots, x_n]$ for F any field.
 - iii) $x^4 + 1 \in F_p[x]$, p an odd prime (note that $p^2 \equiv 1 \pmod{8}$).
 - iv) $x^p + x^{p-1} + \cdots + x + 1 \in F_p[x]$, p an odd prime.
5. Let R be a commutative ring with 1, and let $r_1, \dots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \dots, a_n) \in R^n \mid a_1r_1 + \cdots + a_nr_n = 0\}$, show that M is a projective R module.
6. Let K be a finite Galois extension of \mathbf{Q} with $\text{Gal}(K/\mathbf{Q}) \cong A_4$. How many subfields does K contain, what are their dimensions over \mathbf{Q} , and which are Galois over \mathbf{Q} ?

ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 1996

- (1) Let G be a group with $|G| = 585$. Show that G contains a normal cyclic subgroup of prime index. Describe G up to isomorphism. Show that $Z(G) \neq e$ and has composite order.
- (2) For any $n > 3$, show that each element of the symmetric group S_n is a product of permutations, each having no fixed point in $\{1, 2, \dots, n\}$.
- (3) Show that any cyclic group G with square free order ($a > 0$ and $a^2 \nmid |G|$ implies that $a = 1$) is the Galois group over \mathbb{Q} of some field extension $K \supseteq \mathbb{Q}$. (Hint: For a suitable $n > 0$, consider $x^n - 1 \in \mathbb{Q}[x]$.)
- (4) Let $(\mathbb{Q}, +)$ be the rational numbers under addition.
 - (a) Show that $(\mathbb{Q}, +)$ is not a finitely generated Abelian group.
 - (b) Show that any finitely generated \mathbb{Z} submodule of \mathbb{Q} is free.
 - (c) Determine if $(\mathbb{Q}, +)$ is a free \mathbb{Z} module.
 - (d) Is $(\mathbb{Q}, +)$ a projective \mathbb{Z} module.
- (5) Let $R = \mathbb{C}[x_1, \dots, x_n]$ and let I and J be ideals of R satisfying: for all $\alpha \in \mathbb{C}^n$, $f(\alpha) = 0$ for all $f \in I$ if and only if $g(\alpha) = 0$ for all $g \in J$.
 - (a) Show that $(I + J)/I$ is a nil ring.
 - (b) Show that $(I + J)/I$ is a nilpotent ring. (Note that $(I + J)/I$ is an ideal of R/I .)
- (6) If R is a finite ring and $x^5 = x$ for all $x \in R$, describe the structure of R .

Partial credit is given for partial solutions.

1. Up to isomorphism, describe all group of order 495.
2. Let $x^4 - 7 \in F[x]$ for $F \subseteq \mathbb{C}$. If $F \subseteq M \subseteq \mathbb{C}$ and M is a splitting field for $x^4 - 7$ over F , find $\text{Gal}(M/F)$: when $F = \mathbb{Q}$; when $F = \mathbb{Q}[\sqrt{7}]$; and when $F = \mathbb{Q}[i]$, with $i^2 = -1$.
3. Let M be a finitely generated $F[x]$ module (F a field). If every submodule of M has a complement, describe the structure of M in terms of $F[x]$. (Recall that a submodule H of a module M has a complement if there is a submodule H' so that $M \cong H \oplus H'$; i.e. $H + H' = M$ and $H \cap H' = (0)$.)
4. Show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal of $\mathbb{C}[x, y]$ generated by $x^3 + y^2$ and $y^3 + yx$.
5. Let R be a commutative Noetherian ring with no nonzero nilpotent element. Set $A = \{\text{ann } I \mid I \text{ is a nonzero ideal of } R\}$ and $M = \{\text{maximal elements in } A\}$. Prove that R embeds in a direct sum of finitely many domains as follows:
 - a) Show that the elements of M are prime ideals in R .
 - b) For $P \neq Q$ in M , show $\text{ann } Q \subseteq P$.
 - c) Show that M is finite (consider sums of $\text{ann } P_i$ for $P_i \in M$).
 - d) Show that the intersection of the elements in M is zero.
6. Let R be a finite dimensional algebra over the field F . Assume that for every $r \in R$ there some $g(x) \in F[x]$, depending on r , so that $r + g(r)r^2 = 0$. Determine the structure of R .

Algebra Qualifying, Prequalifying, and M.A. Comprehensive Exam, 11/97

Directions: Work any 5. Partial credit in units of 1/4 is given for partial solutions.

1. Let G be a finite group, N a normal subgroup, and P a p -Sylow subgroup of N .
 - a). Show that $G = NN_G(P)$, $N_G(P) = \{g \in G : gPg^{-1} = P\}$ the normalizer of P in G .
 - b). Let $\Phi(G)$ denote the intersection of the proper maximal subgroups of G . Show that $\Phi(G)$ is normal in G and if H is a subgroup such that $G = \Phi(G)H$ then $H = G$.
 - c). Show that $\Phi(G)N_G(P) = G$ and that every p -Sylow subgroup P of $\Phi(G)$ is normal in G .
2. View the $n \times n$ matrix T over the ring of integers \mathbb{Z} as a linear transformation on \mathbb{Z}^n ; that is, $T(X) = XT$, the matrix product of $X = (x_1, \dots, x_n)$ and T . Set $Im(T) = \{T(X) : X \in \mathbb{Z}^n\}$, $Ker(T) = \{X \in \mathbb{Z}^n : T(X) = 0\}$.
 - a). Show that $\mathbb{Z}^n = Ker(T) \oplus Im(T)$.
 - b). What is the structure of the abelian group $\mathbb{Z}^n/Im(T)$ if $T^2 = pI_n$, p a prime and I_n the $n \times n$ identity matrix.
3. Let $\mathbb{Q} \subset F \subset \mathbb{C}$ where F is the field generated over the rationals \mathbb{Q} by all roots of unity in the field \mathbb{C} of complex numbers. Let $a_1, \dots, a_k \in \mathbb{Q}$, $p_1 < \dots < p_k$ primes, and set $M = F(a_1^{1/p_1}, \dots, a_k^{1/p_k})$.
 - a). Show that M is a Galois extension of F .
 - b). Describe the Galois group of M over F .
 - c). For any subfield $F \subset K \subset M$, show that $K = F(S)$ for some subset S of $\{a_1^{1/p_1}, \dots, a_k^{1/p_k}\}$.
4. Let $\overline{\mathbb{F}}_p$ denote the algebraic closure of the finite field \mathbb{F}_p with p elements, p a prime, and let \mathbb{F}_{p^n} denote the subfield of $\overline{\mathbb{F}}_p$ with p^n elements.
 - a). For $x \in \overline{\mathbb{F}}_p$, show that $x^{\frac{p^n-1}{p-1}} \in \mathbb{F}_p$ if and only if $x \in \mathbb{F}_{p^n}$.
 - b). Let $F(n) = \{x \in \overline{\mathbb{F}}_p : x^n \in \mathbb{F}_p\}$. Show that $F(n)$ is finite and $F(pn) = F(n)$.
5. Let F denote a field, σ, τ automorphisms of F generating an abelian subgroup H of $Aut(F)$ of finite order s . The twisted polynomial ring $F_H[x, y]$ consists of all polynomial expressions $\sum_{i,j=0}^m f_{ij}x^i y^j$, $f_{ij} \in F$ in commuting indeterminates x, y over F subject to the usual polynomial addition and multiplication except that $xf = \sigma(f)x$, $yf = \tau(f)y$. Let Z denote the center of $F_H[x, y]$.
 - a). Show that Z contains $F_0[x^s, y^s]$, F_0 the subfield of F fixed by H .
 - b). Show that $F_H[x, y]$ is Noetherian and Z is Noetherian.
6. Let I denote an ideal in $\mathbb{C}[x_1, \dots, x_n]$, \mathbb{C} the field of complex numbers and suppose that I is the intersection of k maximal ideals. Show that if $k < n$ then I contains a homogeneous linear polynomial $a_1x_1 + \dots + a_nx_n$ with $a_i \neq 0$ for some i , $1 \leq i \leq n$. Give an example to show that this can fail for $k \geq n$.
7. Let R is a finite dimensional, semisimple \mathbb{C} -algebra, \mathbb{C} the field of complex numbers, and for $r \in R - \{0\}$, let $m_r(x) \in \mathbb{C}[x]$ denote the monic polynomial of least degree such that $m_r(r) = 0$; i.e., the minimal polynomial for r . Show that R is commutative if and only if $m_r(x)$ has no multiple roots $\forall r \in R - \{0\}$, and R is noncommutative if and only if $\deg m_r(x) < \dim R/k \forall r \in R - \{0\}$.

ALGEBRA QUALIFYING EXAM MAY, 1998

Partial credit is given for partial solutions.

1. Let G be a group of order 105. Show that G contains a normal subgroup of index 3 and determine how many possibilities there are for the structure of G , up to isomorphism. Show that G has a nontrivial center.
2. For a prime integer p , a group G is called p -divisible if the function $f_p: G \rightarrow G$ given by $f_p(g) = g^p$ is surjective (i.e. onto). If G is Abelian and p -divisible, show that G is finitely generated if and only if G is finite with order relatively prime to p .
3. Let $\mathbf{Q} \subseteq M \subseteq \mathbf{C}$ with M a finite dimensional Galois extension of \mathbf{Q} , the rational numbers. If for all subfields $\mathbf{Q} \subseteq L \subseteq M$, $[L:\mathbf{Q}]$ is even, what can the order of $\text{Gal}(M/\mathbf{Q})$ be? In this case, show that M embeds in a radical extension of \mathbf{Q} .
4. Let $\mathbf{C}[x_1, x_2, \dots, x_n] = R$ and let $f(X) = f(x_1, x_2, \dots, x_n) \in R$ be irreducible. Given $g(X), h(X) \in R$ so that $g(\alpha) - h(\alpha) = 0$ for all $\alpha \in \mathbf{C}^n$ satisfying $f(\alpha) = 0$, show that $g(X) + (f(X)) = h(X) + (f(X))$ in $R/(f(X))$.
5. Let R be a commutative Noetherian ring with 1. Prove that R is isomorphic to a finite direct sum of fields if and only if every (ring) homomorphic image of R is projective as an R module.
6. Let F be a finite field and let A be an F subalgebra of $M_n(F)$.
 - a) If A is a domain, show that $\dim_F A \leq n$.
 - b) If A is simple with $F \cdot I_n$ as its center, show that $\sqrt{\dim_F A}$ is an integer and divides n .

Written Qualifying Exam, Algebra, Nov. 1998

Directions. Partial credit in units of $1/4$ is given for partial solutions.

1. Let G be a group of order $p^a q^b$, p, q distinct primes and a, b positive integers. Prove that if $q < p$ and the order of $q \bmod p$ exceeds b then G is solvable.

2. Let G be a finitely generated abelian group (i.e., a finitely generated \mathbb{Z} -module).

a). Prove that G has no elements of order p , p a prime, if and only if $G \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p^r$ for some positive integer r , \mathbb{Z}_p is the local ring of rational numbers with denominator prime to p .

b). Prove that G is projective if and only if there is an integer r such that $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian groups H .

3. Let \mathbb{F}_{p^n} be a finite field with p^n elements, p a prime. Recall that the norm map $N : \mathbb{F}_{p^n}$ is defined by $N(x) = \prod_{g \in \text{Gal}_{\mathbb{F}_p} \mathbb{F}_{p^n}} g(x)$ and the trace map is defined by $T(x) = \sum_{g \in \text{Gal}_{\mathbb{F}_p} \mathbb{F}_{p^n}} g(x)$.

Determine the image of each of these maps, show that the kernel of the norm map is $\{x/g(x) : x \in \mathbb{F}_{p^n}^\times, g \in \text{Gal}_{\mathbb{F}_p} \mathbb{F}_{p^n}\}$ and that the kernel of the trace map is $\{x - g(x) : x \in \mathbb{F}_{p^n}, g \in \text{Gal}_{\mathbb{F}_p} \mathbb{F}_{p^n}\}$.

4. Let R be a subring of $\mathbb{C}[x_1, \dots, x_n]$ containing \mathbb{C} and assume that the field of quotients of R is $\mathbb{C}(x_1, \dots, x_n)$. Show that there are polynomials $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$ such that $d\mathbb{C}[x_1, \dots, x_n] \subset R$ if and only if $d \in I = f_1\mathbb{C}[x_1, \dots, x_n] + \dots + f_s\mathbb{C}[x_1, \dots, x_n]$. In addition, show that I cannot be a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$.

5. Maschke's theorem implies that the group algebra $k[G]$ over a field k of characteristic zero is semisimple when G is a finite group. Using this fact,

a). Determine the structure of $\mathbb{C}[S_3]$, S_3 the symmetric group on three symbols.

b). An epimorphism of groups, $\phi : G \rightarrow H$, induces an epimorphism $\Phi : k[G] \rightarrow k[H]$ on the corresponding group rings over k . Prove that if k has characteristic 0 and G is finite then $k[H]$ is a ring direct summand of $k[G]$.

6. Determine the galois group of $x^4 - p$ over the rationals, p a prime, and determine all subfields of its splitting field which are normal over the rational numbers.

ALGEBRA QUALIFYING EXAM MAY, 1990 9

Partial credit is given for partial solutions.

1. Let G be a group of order $1705 = 5 \cdot 11 \cdot 31$. Describe the possible structures of G up to isomorphism.
2. Let G be a group and $N \triangleleft G$. Show: i) G is solvable $\Leftrightarrow N$ and G/N are solvable; and ii) if $|G| = p^n$ for p a prime then G is solvable.
3. Let $f(x) \in \mathbf{Q}[x]$ be irreducible with $\deg f = p$, an odd prime, and let $K \subseteq \mathbf{C}$ be a splitting field for $f(x)$ over \mathbf{Q} . Suppose that $f(x)$ has exactly two roots in $\mathbf{C} - \mathbf{R}$. Prove that $\text{Gal}(K/\mathbf{Q}) \cong S_p$.
4. Using methods of algebraic geometry show that there is a fixed $m > 0$ so that for any linear polynomial $f(x,y,z,t) = ax + by + cz + dt$, $f(x,y,z,t)^m \in (x^{19}y^{23}z^{29}t^{31}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subseteq \mathbf{C}[x,y,z,t]$.
5. Let $A = \mathbf{C}[x, \sigma]$ be the twisted polynomial ring over \mathbf{C} where σ is complex conjugation. The elements of A are the polynomials $p(x) = c_n x^n + \dots + c_1 x + c_0$ which add in the usual way but with multiplication given by $xa = \sigma(a)x = \bar{a}x$, and extended by the associative and distributive laws. The general expression for products is $\sum a_i x^i \sum b_j x^j = \sum_{i+j=k} (\sum a_i \sigma^i(b_j)) x^k$.
 - i) Find the center of A .
 - ii) Is the center of A a Noetherian ring (and why)?
 - iii) Show that A is a left and a right Noetherian ring.
6. Let R be a right Artinian ring so that each $r \in R$ satisfies $r^5 = r$.
 - i) Show that R is a finite ring.
 - ii) Show that there is some $m \geq 1$ so that R has exactly 2^m elements satisfying $x^2 = x$.
 - iii) Using m in ii), find the possible values for $|R| = \text{card}(R)$.

ALGEBRA QUALIFYING EXAM NOVEMBER, 1999

Partial credit is given for partial solutions.

1. For p and q distinct primes show that any group of order p^2q is solvable.
2. Let G be a finite Abelian group so that whenever H and K are subgroups of G of the same order then $H \cong K$ as groups. Describe the possible structures of G . If $|G| = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, up to isomorphism how many possibilities are there for G ?
3. Let p_1, \dots, p_k be distinct primes in \mathbf{Z} and set $F = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_k}) \subseteq \mathbf{R}$.
 - i) Show that F is a Galois extension of \mathbf{Q} with $\text{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^k$.
 - ii) Show that $F = \mathbf{Q}(\sqrt{p_1} + \dots + \sqrt{p_k})$.
4. For F a field and $R = F[x_1, \dots, x_n]$ let M be a finitely generated R module. Show that there are positive integers s and t and an exact sequence of R modules $0 \rightarrow K \rightarrow R^s \rightarrow R^t \rightarrow M \rightarrow 0$.
5. If I is a nonzero ideal of $R = \mathbf{C}[x_1, \dots, x_n]$ which is not maximal then if R/I is a domain, show that $\dim_{\mathbf{C}} R/I$ must be infinite.
6. If $R \neq \{0\}$ is a finite ring so that each $r \in R$ satisfies the polynomial $x^8 = x$, describe the possible structures of R .

ALGEBRA QUALIFYING EXAM MAY, 2000

Partial credit is given for partial solutions.

1. Up to isomorphism describe all groups of order 595 (5·7·17).
2. Let M be a finitely generated module over a PID R . If $M \otimes_R M \cong M$ determine the structure of M .
3. Let $\rho \in \mathbb{C}$ be a primitive p^{th} root of 1 for an odd prime p and set $L = \mathbb{Q}(\rho)$.
What is $\text{Gal}(L/\mathbb{Q})$? If m is the number of different positive integer divisors of $p - 1$, how many fields F satisfy $\mathbb{Q} \subseteq F \subseteq L$ and how many of these are Galois extensions of \mathbb{Q} ? What are the $\text{Gal}(F/\mathbb{Q})$? Show that $[L : \mathbb{R} \cap L] = 2$. Show that $N_{L/\mathbb{Q}}(1 - \rho^j) = p$ for any $1 \leq j \leq p-1$.
4. Let R be a commutative Noetherian ring with 1 and let $\varphi: R[x_1, \dots, x_n] \rightarrow R[x_1, \dots, x_n]$ be a surjective ring homomorphism. Show that φ is an automorphism.
5. Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$.
 - i) Show that there is $k > 0$ so that $(\sqrt{I})^k \subseteq I$.
 - ii) Prove that if I is maximal then I/I^k is a finite dimensional \mathbb{C} -vector space for all $k \geq 0$.
 - iii) Show that $\mathbb{C}[x_1, \dots, x_n]/I$ is finite dimensional over $\mathbb{C} \Leftrightarrow \{\alpha \in \mathbb{C}^n \mid f(\alpha) = 0, \text{ all } f \in I\}$ is finite.
6. If R is a finite ring with 1 and $x, y \in R$ satisfy $xy = 1$, show that $yx = 1$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 2000

- (1) Describe all groups of order $3 \cdot 17 \cdot 23$ up to isomorphism.
- (2) Let G be a finitely generated Abelian group so that every proper homomorphic image of G is cyclic. Prove that G is cyclic or that $|G| = p^2$ for p a prime.
- (3) Let $K \subseteq \mathbb{C}$ be a splitting field over \mathbb{Q} of $x^5 - 5$. Describe $\text{Gal}(K/\mathbb{Q})$. Describe those fields $\mathbb{Q} \subseteq M \subseteq K$ with M Galois over \mathbb{Q} , and for these find $\text{Gal}(M/\mathbb{Q})$.
- (4) Let \overline{F} be an algebraic closure of the field F . If $M \subseteq F[x_1, \dots, x_n]$ is a maximal ideal, show that $V(M) = \{\alpha \in \overline{F} \times \dots \times \overline{F} \mid f(\alpha) = 0 \text{ for all } f \in M\}$ is finite and not empty.
- (5) Let $M \subseteq \mathbb{Q}$ be Noetherian \mathbb{Z} -submodule. For N a \mathbb{Z} -submodule of M , show M/N is finite (as a set) $\Leftrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$.
- (6) If R is a right Artinian ring and $x^3 = x$ for all $x \in R$, show: R is commutative; R is finite; and R has $2^a 3^b$ elements for some $a, b \geq 0$.

ALGEBRA QUALIFYING EXAM FEBRUARY 2002

Partial credit is given for partial solutions

1. Describe all groups of order $2 \cdot 31 \cdot 61$ up to isomorphism.
2. Let G be a finite solvable group. If $\langle e \rangle \neq N \triangleleft G$ and N is minimal (for any $H \triangleleft G$ with $H \subseteq N$ either $H = \langle e \rangle$ or $H = N$) show that $N \cong \mathbb{Z}_p^k = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, for p a prime and some $k \geq 1$.
3. For any prime p show that $x^4 + 1 \in \mathbb{F}_p[x]$ cannot be irreducible. (\mathbb{F}_p is the field of p elements. Note that $p^2 \equiv 1 \pmod{8}$ for any odd prime.)
4. Let $f(X) = f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n] = R$ be irreducible. If $g(X), h(X) \in R$ with $g(\alpha) = h(\alpha)$ for all $\alpha \in \mathbb{C}^n$ satisfying $f(\alpha) = 0$, show that the images of $g(X)$ and $h(X)$ in $R/(f(X))$ are equal, that is $g(X) + (f(X)) = h(X) + (f(X))$.
5. Let R be a commutative ring with 1.
 - i) Show that R is a Noetherian ring \Leftrightarrow for each maximal ideal M of R the localization R_M at M is a Noetherian ring.
 - ii) Show that R is a Noetherian \Leftrightarrow every localization of the polynomial ring $R[x, y]$ at its maximal ideals is Noetherian.
6. If R is a right Artinian algebra over the algebraically closed field F show that R is algebraic over F of bounded degree. That is for some fixed $M > 0$ and any $r \in R$, there is some nonzero $f(x) \in F[x]$ depending on r so that $f(r) = 0$ and $\deg f \leq M$.

Σ 10 A/B

Algebra Qualifying Exam September 2002

Partial credit is given for partial solutions

1. Let k be a field and let S_n act on the polynomial ring $k[X_1, \dots, X_n]$ by permuting the variables, i.e. $\sigma \cdot f = f^\sigma$ where

$$f^\sigma(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

Show that for any given f , the number of distinct polynomials of the form f^σ is a divisor of $n!$.

2. Show that there are exactly 2 groups of order $11 \cdot 43^2$.

3. Let $f(X) = X^3 - X - 1 \in \mathbb{Q}[X]$. Find the splitting field K of f over \mathbb{Q} , the Galois group of the extension $\mathbb{Q} \subseteq K$ and give the number of subfields of K of each degree.

4. Let k be an algebraically closed field, and let $R = k[X_1, \dots, X_n]$.

a) Show that if \mathfrak{p} is any prime ideal of R , then \mathfrak{p} is an intersection of maximal ideals.

b) If $n = 2$, show that the ideal $\mathfrak{p} = (X_1 + X_2)$ is prime and describe all maximal ideals \mathfrak{m} such that $\mathfrak{p} \subseteq \mathfrak{m}$.

5. Let R be a commutative algebra over the field k , and $A_1, \dots, A_t \in M_n(R)$ be $n \times n$ -matrices with entries in R . Show that there exists a k -subalgebra $S \subseteq M_n(R)$ containing A_i , $1 \leq i \leq t$, such that S is (left) noetherian. (hint: Try to find a subalgebra $R_0 \subseteq R$ such that $S = M_n(R_0)$.)

6. Let R be a finite ring. Show that if $x, y \in R$ satisfy $xy = 1$, then they also satisfy $yx = 1$.

(hint: First consider the case R is semi-simple.)

Algebra Qualifying Exam February 2003

Partial credit is given for partial solutions

1. Show that there are four non-isomorphic groups of order 18. For partial credit show that there are at most four non-isomorphic groups of order 18.

2. Let D_n be the dihedral group of order $2n$. Recall that D_n is generated by an element a of order n , an element b of order 2, and the relation $bab = a^{-1}$ holds.

i) Show that the derived group D'_n is:

$$D'_n = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ odd;} \\ 2\mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ even.} \end{cases}$$

(Hint: Show first that $a^2 \in D'_n$.) Conclude that D_n is solvable.

ii) Show that: D_n is nilpotent if and only if n is a power of 2. (Hint: Show that for $i > 1$, $\gamma_i(D_n) = \langle a^{2^{i-1}} \rangle$, where γ_i is the lower central series.)

3. Let $f(X) \in K[X]$ be an irreducible polynomial of degree 4 and G its Galois group. Show that G is isomorphic to one of the following groups: $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_4 , A_4 , or S_4 .

4. Let $f(X) = (X^3 - 2)(X^2 - 3) \in \mathbb{Q}[X]$. Find the splitting field K of f over \mathbb{Q} , the Galois group of the extension $\mathbb{Q} \subseteq K$ and give the number of subfields of K of each degree.

5. i) How many different similarity classes of 4×4 nilpotent matrices are there over a field F ?

ii) What is the characteristic polynomial of such a matrix?

iii) Prove or give a counterexample: If two such matrices have the same minimum polynomial and characteristic polynomial, they are similar.

6. Let $R = \mathbb{Q}[x, y, z]$.

i) Prove that every simple R -module M is finite dimensional over \mathbb{Q} .

ii) Show that there is no bound on the dimension of a simple module.

7. Consider a homogeneous system of m linear equations in n variables over \mathbb{Z} and assume that $d = n - m \geq 0$.

i) Show that the set of solutions is a free abelian group of rank at least d .

ii) What is the rank of the group of solutions (in terms of the coefficient matrix of the system of equations)?

Algebra Qualifying Exam January 2004

- (1) Determine the number of nonisomorphic groups of order $2 \cdot 7 \cdot 17 \cdot 23$.
- (2) Let G be a finite p -group, p a prime. Prove that the following are equivalent:
 - (a) G does not contain a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$;
 - (b) Every abelian subgroup of G is cyclic;
 - (c) G has a unique subgroup of order p .
- (3) Let K be a splitting field of $x^4 - 2$ over the rational numbers \mathbb{Q} .
 - (a) Find $[K : \mathbb{Q}]$ and describe the Galois group of K/\mathbb{Q} .
 - (b) How many intermediate fields are normal (Galois) over \mathbb{Q} ? Explain.
- (4) Let k be a commutative and let R, S be commutative k algebras such that R is noetherian and S is a finitely generated k -algebra. Prove that $R \otimes_k S$ is a noetherian ring.
- (5) Let k be a field, B a finitely generated k -algebra and let A be a k -subalgebra of B .
 - (a) If M is a maximal ideal of B , prove that $M \cap A$ is a maximal ideal of A .
 - (b) Give an example to show that this is false if B is not finitely generated.
- (6) Let A be a 5 dimensional algebra over the field k of p elements, p a prime. Assume that for each nonzero $a \in A$, there exists $b \in A$ with $ab = e = e^2 \neq 0$. Find all such algebras up to isomorphism (note that this condition is satisfied by $M_n(F)$ for any field F and you may use this fact).
- (7) Let F be a field and $F[x]$ the polynomial ring over F . Let M be a finitely generated free module over $F[x]$. Let $N_i, i = 1, 2, \dots$ be a descending chain of $F[x]$ -submodules of M . Prove that there exists a positive integer t so that for $i > t$, N_i/N_{i+1} is finite dimensional over F (note that $F[x]/I$ is finite dimensional over F for any nonzero ideal I).

ALGEBRA EXAM SEPTEMBER 2004

Do as many problems as you can

1. Up to isomorphism describe all groups of order $399 = 3 \cdot 7 \cdot 19$. For each group find the order of its center and the order of its commutator subgroup.
2. Suppose R is a finite dimensional algebra over a field F with 1 and $U(R)$, the group of units in R , is abelian. Show that the Jacobson radical $J(R)$ and $R/J(R)$ are commutative.
3. Let L be a subfield of the finite field K of characteristic p . Let $\alpha \in K$ with minimal polynomial $v(x)$ of degree d over L . Show that $v(x)$ splits over K and that for some $q = p^m$ the roots of $v(x)$ in K are $\{\alpha, \alpha^q, \dots, \alpha^{q^{d-1}}\}$.
4. Let R be a commutative ring with 1 and M, N, V all R -modules.
 - (a) If M and N are projective show that $M \otimes_R N$ is also a projective R -module.
 - (b) Let

$$\text{Tr}(V) = \left\{ \sum_{i=1}^n \phi_i(v_i) \mid \phi_i \in \text{Hom}_R(V, R), v_i \in V, n = 1, 2, \dots \right\}.$$

If $1 \in \text{Tr}(V)$ show that up to isomorphism some finite direct sum V^k contains R as an R -module direct summand.

5. Show that any surjective ring homomorphism $f : R \rightarrow R$ of a left Noetherian ring R must be an isomorphism. Give an example to show this may be false if the ring is not noetherian.
6. In $\mathbb{C}[x, y]$ show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal $(x^3 + y^2, y^3 + xy)$.

ALGEBRA QUALIFYING EXAM FALL 2005

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let G be a group with $|G| = p^n q^m$ for $p < q$ primes and assume that the order of $[p]_q$ in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ is larger than n . Show that there are subgroups $\langle e \rangle \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{n+m} = G$ with each $H_j \triangleleft H_{j+1}$ and H_{j+1}/H_j cyclic of prime order.
2. Let $F \subseteq L$ be finite fields with $[L : F] = 3$. If $\alpha \in F$ show that there is $\beta \in L$ satisfying $\beta^3 = \alpha$.
3. If $p(x) = x^8 + 6x^4 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over \mathbb{Q} , argue that $\text{Gal}(M/\mathbb{Q})$ is solvable.
4. Let R be a commutative ring with 1 and let $x_1, \dots, x_n \in R$ so that $x_1 y_1 + \cdots + x_n y_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \dots, r_n) \in R^n \mid x_1 r_1 + \cdots + x_n r_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that A has n generators as an R module, and that when $R = F[x]$ for F a field then A_R is free of rank $n - 1$.
5. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and let I be a nonzero proper ideal of R . If $A \in M_k(R)$ and $A(\alpha) = 0_{k \times k}$ for all $\alpha \in \text{Var}(I)$, show that for some $s > 0$, $A^s \in M_k(I)$.
6. If R is a right Artinian algebra over \mathbb{C} , show there is an integer $m \geq 1$ so that if $x \in R$ and $x^k = 0$ for some integer $k \geq 1$, then $x^m = 0$: that is, the indices of nilpotence of the nil elements of R are bounded.

ALGEBRA QUALIFYING EXAM FALL 2005

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let G be a group with $|G| = p^n q^m$ for $p < q$ primes and assume that the order of $[p]_q$ in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ is larger than n . Show that there are subgroups $\langle e \rangle \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{n+m} = G$ with each $H_j \triangleleft H_{j+1}$ and H_{j+1}/H_j cyclic of prime order.
2. Let $F \subseteq L$ be finite fields with $[L : F] = 3$. If $\alpha \in F$ show that there is $\beta \in L$ satisfying $\beta^3 = \alpha$.
3. If $p(x) = x^8 + 6x^4 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over \mathbb{Q} , argue that $\text{Gal}(M/\mathbb{Q})$ is solvable.
4. Let R be a commutative ring with 1 and let $x_1, \dots, x_n \in R$ so that $x_1 y_1 + \cdots + x_n y_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \dots, r_n) \in R^n \mid x_1 r_1 + \cdots + x_n r_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that A has n generators as an R module, and that when $R = F[x]$ for F a field then A_R is free of rank $n - 1$.
5. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and let I be a nonzero proper ideal of R . If $A \in M_k(R)$ and $A(\alpha) = 0_{k \times k}$ for all $\alpha \in \text{Var}(I)$, show that for some $s > 0$, $A^s \in M_k(I)$.
6. If R is a right Artinian algebra over \mathbb{C} , show there is an integer $m \geq 1$ so that if $x \in R$ and $x^k = 0$ for some integer $k \geq 1$, then $x^m = 0$: that is, the indices of nilpotence of the nil elements of R are bounded.

ALGEBRA QUALIFYING EXAM SPRING 2006

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions. Let \mathcal{Q} denote the field of rational numbers, \mathcal{C} the field of complex numbers, and \mathbf{F}_q the finite field of q elements.

- Up to isomorphism, describe the groups of order $3 \cdot 17 \cdot 19$.
- For p a prime, $q = p^k$, and n a positive integer, describe a condition that guarantees that the multiplicative group $\mathbf{F}_q^* = (\mathbf{F}_q - \{0\}, \cdot)$ contains an element of order n .
 - Determine the cardinality of a splitting field L over \mathbf{F}_3 of $x^{13} - 1 \in \mathbf{F}_3[x]$, and the structure of $\text{Gal}(L/\mathbf{F}_3)$
- Let $f(x) = (x^3 - 2)(x^3 - 3) \in \mathcal{Q}[x]$, $M \subseteq \mathcal{C}$ a splitting field over \mathcal{Q} for $f(x)$, $G = \text{Gal}(M/\mathcal{Q})$, and $\omega \in \mathcal{C}$ a primitive cube root of unity.
 - Show that $\omega \in M$.
 - Assume that $3^{1/3} \notin \mathcal{Q}(\omega, 2^{1/3}) \subseteq M$, and use this to find the order of G .
 - Describe how the elements of G act on M .
 - Determine the structure of G .
- In $\mathcal{C}[x, y]$ show that for some integer $m \geq 1$, $(3x^2 + 10xy + 3y^2)^m \in (x + y - 2, x^2 + y^2 - 10)$, the ideal of $\mathcal{C}[x, y]$ generated by $x + y - 2$ and $x^2 + y^2 - 10$.
- Let $g_1, g_2, \dots, g_m, \dots \in R$, a commutative Noetherian ring with 1, and let I be an ideal of R . Assume that for each i there is $k_i \geq 1$ so that $g_i^{k_i} \in I$. Show that there is a positive integer K so that $g_{i_1} g_{i_2} \cdots g_{i_K} \in I$ for any choices of $g_{i_j} \in \{g_i\}$.
- Let S be a finite ring so that for each $x \in S$, $x^5 = x$.
 - Show that S contains no nonzero nilpotent element.
 - Show that, up to isomorphism, S is a direct sum of copies of \mathbf{F}_2 , \mathbf{F}_3 , and \mathbf{F}_5 .

ALGEBRA QUALIFYING EXAM
September, 2006

1. (a) Find the number of Sylow p -subgroups of the symmetric group S_p . Here p is a prime.

(b) Use (a) to prove that

$$(p-1)! + 1 \equiv 0 \pmod{p}.$$

2. Let G be a finite solvable group and H a minimal (non-trivial) normal subgroup. Show H is isomorphic to a direct sum of cyclic groups of order p , for some prime p . (Hint: First show that the commutator subgroup H' of H is $\{e\}$.)

3. Let m_1, m_2, \dots, m_n be positive integers which are pairwise relatively prime (that is, $\gcd(m_i, m_j) = 1$ for all $i \neq j$).

(a) Show that $F := \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \dots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \dots, \sqrt{m_n})$. (Hint: induction.)

(b) Show that $F = \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \dots + \sqrt{m_n})$ is Galois over \mathbb{Q} . What is its Galois group?

4. Let k be a field, let R be a commutative k -algebra, and let $S = M_n(R)$ be all $n \times n$ -matrices with entries in R . Choose $A_1, \dots, A_m \in S$. Show that there exists a (left) Noetherian k -subalgebra S_0 of S which contains all of the matrices A_i . (Hint: Consider the subalgebra $R_0 \subset R$ generated by all entries of $A_1, \dots, A_m \in S$.)

5. Let $R = \mathbb{C}[x, y, z]$, let $I = (x^2z^3 - y^2z + xyz - x^2y)$ be an ideal of R , and define $S = R/I$.

(a) Prove that the polynomial $x^2z^3 - y^2z + xyz - x^2y$ is irreducible in R . (Hint: consider it as an element of $\mathbb{C}[x, y][z]$.)

(b) Show that S is a Noetherian integral domain.

(c) Prove that in S the intersection of all maximal ideals is $\{0\}$.

6. Let k be a finite field and let R be a finite dimensional semi-simple k -algebra such that for all $r \in R$, there exists a positive integer $n = n(r) > 0$ such that $r^{n(r)}$ is in the center of R . Prove that R is commutative.

7. Let M be a finitely generated \mathbb{Z} -module with torsion submodule $T(M)$.

(a) Justify: $M/T(M)$ is a free \mathbb{Z} -module.

For parts (b) and (c), set $r(M) := \text{rank}(M/T(M))$.

(b) Show that $r(M) = \dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$.

(c) Assume that

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

is an exact sequence of \mathbb{Z} -modules. Show that

$$r(N) = r(M) + r(P).$$

ALGEBRA PH.D QUALIFYING EXAM SPRING 2007

- For G a finite group with $|G| > 1$ and p a prime dividing the order of G , let $O_p(G) = \bigcap \{P \in \text{Syl}_p(G)\}$.
 - Show that $O_p(G)$ is a normal subgroup of G .
 - Show that if N is a normal subgroup of G with $|N| = p^k$, then $N \subseteq O_p(G)$.
 - Prove that if G is solvable then for some p , $|O_p(G)| \neq 1$.
- Let $F = GF(p^n)$ be a field of (exactly) p^n elements. Suppose that k is a positive integer dividing n , and set $B = \{a^{p^k} + a^{p^{2k}} + \cdots + a^{p^n} \mid a \in F\}$.
 - Show that $B \subseteq E$, a subfield of F with p^k elements.
 - Show that $B = E$.
- Let $A \in M_n(\mathbf{Q})$ with $A^k = I_n$. If j is a positive integer with $(j, k) = 1$, show that $\text{tr}(A) = \text{tr}(A^j)$. (Hint: Consider $A \in M_n(\mathbf{Q}(\varepsilon))$ for $\varepsilon = e^{2\pi i/k}$, where $\varepsilon^2 = -1$.)
- Let R be a commutative ring with 1 and let M be a Noetherian R -module. If $f \in \text{Hom}_R(M_R, M_R)$ is surjective, show that f is an automorphism of M_R .
- Let $f, g \in \mathbf{C}[x, y]$ so that $(0, 0) \in \mathbf{C}^2$ is the only common zero of f and g . Prove that there is a positive integer m so that whenever $h \in \mathbf{C}[x, y]$ has no monomial of degree less than m , then $h \in f \cdot \mathbf{C}[x, y] + g \cdot \mathbf{C}[x, y]$.
- For a fixed positive integer $n > 1$, describe all finite rings R so that $x^n = x$ for all $x \in R$.

ALGEBRA EXAM FALL 2007

1. Let G be a group of order 105.
 - (a) Show G has a normal subgroup of index 3.
 - (b) Show $Z(G) \neq 1$.
 - (c) Determine all possibilities for G .
2. Let p be a prime. A group G is called p -divisible if the map $x \rightarrow x^p$ is surjective. Suppose that G is a finitely generated abelian group. Show that G is p -divisible if and only if G is finite and p does not divide the order of G .
3. Let $R = \mathbb{C}[x_1, \dots, x_n]$. Suppose that $f \in R$ is irreducible. If $g(a) = h(a)$ whenever $f(a) = 0$, show that $g + (f) = h + (f)$ in $R/(f)$.
4. Let F be a field. Suppose that A is an F -subalgebra of $M_n(F)$ containing the identity of $M_n(F)$.
 - (a) If A is a domain, show that A is a division algebra and $\dim A \leq n$.
 - (b) If A is simple, show that $(\dim A) | n^2$ (hint: Let V be the space of column vectors of size n over F – this is a left $M_n(F)$ -module of dimension n ; show that V is a direct sum of say s isomorphic copies of a simple A -module U . Relate the dimension of A and the dimension of U).
5. Let p be a prime. Let $F := \mathbb{F}_{p^n}$ be the field of size of p^n . Let $f(x) \in F[x]$ be irreducible of degree t .
 - (a) Show that the splitting field for f has size p^{nt} .
 - (b) If $n = 1$, show that $f(x) | (x^{p^m} - x)$ if and only if $t | m$.
 - (c) How many irreducible polynomials of degree 6 are there over \mathbb{F}_2 ?
6. Let R be a commutative ring with 1. Assume that $R = a_1R + \dots + a_nR$ for some $a_i \in R$. Let $M =: \{(r_1, \dots, r_n) \in R^n \mid \sum a_i r_i = 0\}$. Show that M is a projective R -module and can be generated by n elements as an R -module.

Algebra Qualifying Examination, Spring 2008

Directions

This exam consists of 7 problems. Please do 6 of them and show your work. If you are using a well-known result in your proof, please refer to it by name. Good Luck!

1. Let G be a finite group with A a normal subgroup of G . Let P be a Sylow p -subgroup of A . (a) If $g \in G$, show that $gPg^{-1} = xPx^{-1}$ for some $x \in A$. (b) Prove that $G = AN_G(P)$. (c) Prove that if $[G : A] = p$, then the number of Sylow p -subgroups of G that contain A is equal to $[N_G(P) : N_G(Q)]$ where Q is a Sylow p -subgroup of G containing P .

2. Let G be a finite group of order n . Let $f : G \rightarrow S_n$ be the regular representation of G . (a) Show that the image of f is contained in the alternating group if and only if the Sylow 2-subgroup of G is not cyclic. (b) Use (a) to show that if G has a cyclic nontrivial Sylow 2-subgroup, then G contains a normal subgroup N of odd order (hint: show first that G has a normal subgroup of index 2).

3. Let A be a (commutative) integral domain that is not a field and let K be the quotient field of A .

(a) Prove that K is not finitely generated as an A -module. Hint: use Nakayama's lemma.

(b) Can K ever be finitely generated as an A -algebra?

4. Let K be a field and p be a prime number that is not equal to the characteristic of K . Assume that K does not contain a primitive p -th root of unity. Let $a \in K^*$ that is not a p -th power and let b be an

element of an separable closure of K with $b^p = a$. Consider the field $L = K(\zeta, b)$, where ζ is a primitive p -th root of unity. Prove that L/K is a Galois extension and that $\text{Gal}(L/K)$ is isomorphic to the group of 2 by 2 matrices

$$\begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$$

where $r \in \mathbb{Z}/p\mathbb{Z}$ and $s \in (\mathbb{Z}/p\mathbb{Z})^*$. Is this group abelian?

5. Let $R = \mathbb{C}[x, y]$ and consider the two ideals $I = (2x + y)$ and $J = (x^2 - y)$. (a) show that I and J are both prime ideals of R , and that each of them is the intersection of all of the maximal ideals containing it.

(b) Consider the ideal $I + J$. Is it a prime ideal?

(c) Same question for $I \cap J$.

(hint: you can get a lot of intuition for this problem by thinking about the analogous varieties in \mathbb{R}^2).

6. (i) Let A be a finitely generated abelian group. If ℓ is a prime number, let $A[\ell]$ denote the set of elements of A that are killed by ℓ . Then $A[\ell]$ and $A/\ell A$ are finite groups, say of orders n_1 and n_2 , respectively. Express the difference $n_2 - n_1$ in terms of other invariants of A .

(ii) If A is an abelian group such that the groups $A[\ell]$ and $A/\ell A$ are finite, is A necessarily finitely generated? Give a proof or a counterexample.

7. Let R be a (left) Artinian ring which is an algebra over the field k . Assume that every $r \in R$ is algebraic over k , with a minimal polynomial of the form

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0,$$

such that either a_0 or a_1 is non-zero. Show that R is a direct sum of division rings.

ALGEBRA QUALIFYING EXAM, Fall 2008

- Let p, q be odd primes with $p > 7$ and $q > 8p$. Let G be a group of order $8pq$.
 - Show that G has a normal subgroup of order pq .
 - Show that G has a normal subgroup of index 2.
 - Show that G has a nontrivial center.
- Let $G = L_1 \times \dots \times L_t$, for $t > 1$, where all of the L_i are simple groups.
 - Assuming that all of the L_i are nonabelian, prove that the only normal subgroups of G are direct products of some subset of the L_i . (Hint: Let N be a normal subgroup of G and show that if the i th projection of N into L_i is nontrivial, then N contains L_i .)
 - Now suppose that all $L_i \cong L$, with L simple (possibly abelian). Show that there is no nontrivial proper subgroup of G which is invariant under all automorphisms of G . (Hint: Consider the abelian and nonabelian cases separately.)
 - Suppose that $G = L \times L$ with L a nonabelian simple group. Let $D = \{(x, x) | x \in L\}$ be the diagonal subgroup. Show that D is a maximal subgroup of G .
- Consider $f(x) = x^4 + x^2 + 9 \in \mathbb{Q}[x]$.
 - Show that $f(x)$ is irreducible over \mathbb{Q} . (Hint: first show that the only possible factors are quadratic, and then see what happens when x is replaced by $-x$.)
 - Find the Galois group of $f(x)$ over \mathbb{Q} .
 - Describe the splitting field of f over \mathbb{Q} and the intermediate fields.
- Let R be a commutative Noetherian ring. Show that any surjective ring endomorphism $\phi: R \rightarrow R$ is an automorphism.
(Hint: consider the iterations $\phi, \phi^2, \phi^3, \dots$)
- Let I be the ideal
$$I = (x^{37}y^{31}z^{29}t^{23}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subset \mathbb{C}[x, y, z, t].$$
If $f(x, y, z, t)$ is any polynomial without constant term show that some power of f is in I .
- Let A be a finite-dimensional algebra over \mathbb{C} . Show that if $x, y \in A$ such that $xy = 1$, then also $yx = 1$.
- Let A, B, C be finitely generated modules over a PID R . Show that B is isomorphic to C if and only if $A \oplus B$ is isomorphic to $A \oplus C$.

ALGEBRA QUALIFYING EXAM, Spring 2009

Throughout, \mathbb{Z} denotes the integers, \mathbb{Q} the rational numbers, \mathbb{R} the real numbers, and \mathbb{C} the complex numbers.

1. Let G be a finite group. Define the *Frattini subgroup* of G to be $\Phi(G)$, the intersection of all maximal subgroups of G .

- (1) Show that $\Phi(G)$ is characteristic in G (i.e. invariant under any automorphism of G).
- (2) Show that if $G = \langle \phi(G), S \rangle$ for some subset S of G , then $G = \langle S \rangle$.
- (3) Let P be a Sylow p -subgroup of $\phi(G)$. Show that P is normal in G (hint: first show that $G = \Phi(G)N_G(P)$ by using Sylow's theorems and then use (2)).
- (4) Show that $\Phi(G)$ is nilpotent.

2. Let G be a finite group acting on the finite set X with $|X| = n > 1$, and suppose that G has N orbits on X . If $g \in G$, let $F(g)$ be the number of $x \in X$ fixed by g .

- (1) Prove that $\sum_{g \in G} F(g) = |G|N$ (this is known as *Burnside's Lemma*).
- (2) Prove that if G is transitive on X , then $F(g) = 0$ for some $g \in G$ (either use (1) or prove directly).
- (3) Show that this is not always true if G is not transitive on X .

3. Let $f(x) = x^4 - x^3 + x^2 - x + 1 \in \mathbb{Q}[x]$. Find the splitting field (over \mathbb{Q}) of $f(x)$, and compute $Gal(K/\mathbb{Q})$.

4. Construct an example of each of the following (with reasons):

- (1) A field extension $F \subsetneq K$ which is normal but not separable.
- (2) A field extension $F \subsetneq K$ which is separable but not normal.
- (3) A field extension $F \subsetneq K$ which is neither separable nor normal.

5. Let F be the field of p elements. Let $A \in G := GL(n, F)$.

- (1) Show that A has order a power of p if and only if $(A - I)^n = 0$.
- (2) Show that if this is the case then the order of A is less than np .
- (3) Show that any such A is similar to an upper triangular matrix.

6. Let M be a finitely generated abelian group, and N a subgroup. If $M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$, show that M/N is torsion.

CONTINUED \rightarrow

7. Consider the polynomial ring $\mathbb{C}[x, y]$ and let I be the ideal $I = (x + y - 2, x^2 + y^2 - 10)$.

- (1) Show that there exists some $m > 0$ such that $(3x^2 + 10xy + 3y^2)^m \in I$.
- (2) Show that the two ideals $I_1 = (x + y - 2)$ and $I_2 = (x^2 + y^2 - 10)$ are prime ideals. Are they maximal?
- (3) Can I be written as an intersection of maximal ideals? Why or why not?

8. Let A be a finite-dimensional algebra over \mathbb{R} , with center $Z = Z(A)$ and Jacobson radical $J = J(A)$. Assume that for any $a \in A$, there is some $n = n(a) \geq 1$ such that $a^{2^n} - a \in Z$.

- (1) Show that $J \subseteq Z$.
- (2) Show that A/J is commutative.

In fact A itself is commutative, although you do not have to show this.

ALGEBRA QUALIFYING EXAM, Fall 2009

Notation: \mathbb{Q} denotes the rational numbers, \mathbb{R} the real numbers, \mathbb{C} the complex numbers, and \mathbb{F}_p the field with p elements, for p a prime.

- Determine up to isomorphism all groups of order $1005 = 3 \cdot 5 \cdot 67$.
- (a) Let G be a group of order $2^m k$, where k is odd. Prove that if G contains an element of order 2^m , then the set of all elements of odd order in G is a (normal) subgroup of G .
(Hint: consider the action of G on itself by left multiplication Φ_L , and then consider the structure of the permutations $\Phi_L(x)$, for $x \in G$.)
(b) Conclude from (a) that a finite simple group of even order must have order divisible by 4.
- Give a brief argument or a counterexample for each statement:
(a) $x^{2^n} + 1 \in \mathbb{Q}[x]$ is irreducible for all positive integers n ;
(b) Any splitting field for $x^{13} - 1 \in \mathbb{F}_3[x]$ has 3^{12} elements. (c) $\text{Gal}(L/\mathbb{Q})$ for L a splitting field over \mathbb{Q} of $x^5 - 2 \in \mathbb{Q}[x]$ has a normal 5-Sylow subgroup.
- Let A denote the commutative ring $\mathbb{R}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 + 1)$.
(a) Prove that A is a Noetherian domain.
(b) Give an infinite family of prime ideals of A that are not maximal.
- Let $R = \mathbb{C}[x_1, \dots, x_n]$, let $A = [a_{ij}] \in M_n(\mathbb{C})$, and choose $b_1, \dots, b_n \in \mathbb{C}$. For each $i = 1, \dots, n$, set $L_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - b_i \in R$, and consider the ideal $I = (L_1, \dots, L_n) \subseteq R$.
Prove that R/I is finite-dimensional \iff the matrix A is invertible in $M_n(\mathbb{C})$.
- Let $R = K[x]$, for K a field, and let M be a finitely-generated torsion module over R . Prove that M is a finite-dimensional K -module.
- Let G be a finite group and K a field, and consider the group algebra $R = KG$ (that is, R is a K -vector space with basis $\{g \in G\}$, and multiplication determined by the group product $g \cdot h$, for $g, h \in G$).
If G is the dihedral group of order 8, find the dimensions of all of the simple (left) modules for $R = \mathbb{F}_5 G$.
(Hint: remember that KG always has the "trivial representation" $V_0 = Kv$, such that for any $g \in G$, $a \in K$, $ag \cdot v = av$.)

Algebra Qualifying Exam -Spring 2010

1. Let $f(x) = x^5 + 3 \in \mathbb{Q}[x]$. Show that the Galois group of f is S_5 .
2. (a) Let G be a group of order pqr , where $p < q < r$ are primes. Show that G contains a normal subgroup of index p .
(b) Determine up to isomorphism all groups of order $3 \cdot 7 \cdot 13$.
3. Let R be a commutative Noetherian ring, and let I, J , and K be ideals of R . We say I is irreducible if $I = J \cap K \Leftrightarrow I = J$, or $I = K$.
(a) Show that every ideal of R is a finite intersection of irreducible ideals.
(b) Show that every irreducible ideal is primary. (An ideal I of R is primary if $R/I \neq 0$, and every zero-divisor in R/I is nilpotent.)
4. Let A be a finite-dimensional algebra over a field K , such that for every $a \in A$, $a^7 = a$. Show that A is a direct product (sum?) of fields. Which fields can arise?
5. Let G and H be finitely generated abelian groups such that $G \otimes_{\mathbb{Z}} H = 0$. Show that G and H are finite and have relatively prime orders.
6. Let S and T be diagonalizable endomorphisms of a finite dimensional complex vector space. If S and T commute show that they are polynomials in each other.
7. What are the prime ideals of $\mathbb{Z}[x]$? What are the maximal ideals? Carefully explain your answers.

ALGEBRA QUALIFYING EXAM FALL 2010

Do all six problems. Each problem is worth 4 points and partial credit may be awarded.

1. Use Sylow's Theorems to show that any group of order $(99^2 - 4)^3$ is solvable.
2. For any finite group G and positive integer m , let $n_G(m)$ be the number of elements g of G that satisfy $g^m = e_G$. If A and B are finite abelian groups so that $n_A(m) = n_B(m)$ for all m , show that as groups $A \cong B$.
3. If $g(x) = x^5 + 2 \in \mathbf{Q}[x]$, for \mathbf{Q} the field of rational numbers, compute the Galois group of a splitting field L over \mathbf{Q} of $g(x)$. How many subfields of L containing \mathbf{Q} are Galois over \mathbf{Q} ?
4. Let P be a minimal prime ideal in the commutative ring R with 1; that is, if Q is a prime ideal in R and if $Q \subseteq P$, then $Q = P$. Show that each $x \in P$ is a zero divisor in R .
5. Set $R = \mathbf{C}[x_1, \dots, x_n]$ with $n \geq 3$ and \mathbf{C} the field of complex numbers. For any subset $S \subseteq R$, let $\mathcal{V}(S) = \{\alpha \in \mathbf{C}^n \mid g(\alpha) = 0 \text{ for all } g \in S\}$. Consider the ideal I of R defined by $I = (x_1 \cdots x_{n-1} - x_n, x_1 \cdots x_{n-2}x_n - x_{n-2}, \dots, x_2 \cdots x_n - x_1)$, so the generators of I are obtained by subtracting each x_j from the product of the others. Show that there are fixed positive integers s and t so that for each $0 \leq i \leq n$, $(x_i^s - x_i)^t \in I$. (Hint: Consider the product of the generators of I .)
6. Let R be a right artinian algebra over an algebraically closed field F . Show that R is algebraic over F of bounded degree. That is, show there is a fixed positive integer m so that for any $r \in R$ there is a nonzero $g_r(x) \in F[x]$ with $g_r(r) = 0$ and with $\deg g \leq m$.

ALGEBRA EXAM FEBRUARY 2011

1. Let G be a finite group with a cyclic Sylow 2-subgroup S .
 - (a) Show that any element of odd order in $N_G(S)$ centralizes S .
 - (b) Show that $N_G(S) = C_G(S)$.
 - (c) Give an example to show that (a) can fail if S is abelian.

2. Let G be a finite group with a cyclic Sylow 2-subgroup $S \neq 1$.
 - (a) Let $\rho : G \rightarrow S_n$ be the regular representation with $n = |G|$. Show that $\rho(G)$ is not contained in A_n .
 - (b) Show that G has a normal subgroup of index 2.
 - (c) Show that the set of elements of odd order in G form a normal subgroup N and $G = NS$.

3. For a group G and p a prime let $G(p) = \{g \in G \mid g^p = 1\}$.
 - (a) Show that if G is Abelian, then $G(p)$ is a subgroup of G . Give an example to show that $G(p)$ need not be a subgroup in general.
 - (b) Let G, H be finitely generated Abelian groups with $G/G(p) \cong H/H(p)$ and $G/G(q) \cong H/H(q)$ for different primes p, q . Show that $G \cong H$.

4. Let R be a prime ring with only finitely many right ideals.
 - (a) Show that R is a simple ring.
 - (b) Prove that either R is finite or R is a division ring.

5. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and let J be a nonzero proper ideal of R . Let $A = A(X), B = B(X) \in M_r(R)$ and assume that $\det(A)$ is a product of distinct monic irreducible polynomials in R . Assume that for each $\alpha = (a_1, \dots, a_n) \in \mathbb{C}^n$, $B(\alpha) \in M_r(\mathbb{C})$ invertible implies that $A(\alpha)$ is invertible. Show that $\det(A)$ divides $\det(B)$ in R .

6. Let L be a splitting field over \mathbb{Q} for $p(x) = x^{10} + 3x^5 + 1$. Let $G = \text{Gal}(L/\mathbb{Q})$.
 - (a) Show that G has a normal subgroup of index 2.
 - (b) Show that 4 divides $|G|$.
 - (c) Show that G is solvable.

ALGEBRA QUALIFYING EXAM FALL 2011

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of integers is \mathbf{Z} , the set of rational numbers is \mathbf{Q} , and set of the complex numbers is \mathbf{C} .

1. Let I and J be ideals of $R = \mathbf{C}[x_1, x_2, \dots, x_n]$ that define the same variety of \mathbf{C}^n . Show that for any $x \in (I + J)/I$ there is $m = m(x) > 0$ with $x^m = 0_{R/I}$. Show there is an integer $M > 0$ so that for any $y_1, y_2, \dots, y_M \in (I + J)/I$, $y_1 y_2 \cdots y_M = 0_{R/I}$.
2. If $K \subseteq L$ are finite fields with $|K| = p^n$ and $[L : K] = m$ then show that for each $1 \leq t < nm$, any $a \in L - K$ has a p^t -th root in L . When $m = 3$, show that every $b \in K$ has a cube root in L .
3. Let F be an algebraically closed field and A an F -algebra with $\dim_F A = n$. If every element of A is either nilpotent or invertible, show that the set of nilpotent elements of A is an ideal M of A , that M is the unique maximal ideal of A , and that $\dim_F M = n - 1$.
4. Let M be a finitely generated $F[x]$ module, for F a field.
 - i) Show that if $f(x)m = 0$ for $f(x) \neq 0$ forces $m = 0$, then M is a projective $F[x]$ module.
 - ii) If H is an $F[x]$ submodule of M show that $M = H \oplus K$ for a submodule K of M if and only if: $f(x)m \in H$ for $f(x) \neq 0$ implies that $m \in H$.
5. Up to isomorphism, describe the possible structures of any group of order $987 = 3 \cdot 7 \cdot 47$.
6. Let $R = \mathbf{Z}[x_1, x_2, \dots, x_n, \dots]$ and let $\{f_i(X) \mid i \geq 1\} \subseteq R$ satisfy $f_1(X)R \subseteq f_2(X)R \subseteq \cdots \subseteq f_i(X)R \subseteq \cdots$. Show $f_s(X)R = f_m(X)R$ for some m and all $s \geq m$.
7. Let U be the set of all n -th roots of unity in \mathbf{C} , for all $n \geq 3$, and set $F = \mathbf{Q}(U)$. For primes $p_1 < \cdots < p_k$ and nonzero $a_1, \dots, a_k \in \mathbf{Q}$, set $M = F(a_1^{1/p_1}, \dots, a_k^{1/p_k}) \subseteq \mathbf{C}$. Show that M is Galois over F with a cyclic Galois group. For any subfield $F \subseteq L \subseteq M$, show that there is a subset T of $\{a_j^{1/p_j}\}$ so that $L = F(T)$.